ELEMENTARY MENSURATION.

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ELEMENTARY

MENSURATION

BY

F. H. STEVENS, M.A.
SOMETIME A MASTER IN CLIFTON COLLEGE.

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PREFACE

This Text-book of Elementary Mensuration is intended to meet the needs of two classes of learners. It accordingly includes a first and second course. The first course provides for those whose knowledge of Geometry is confined to Euclid's First Book, and of Algebra to the meaning of the simplest symbols: while in the second course somewhat more difficult questions are offered to students who have mastered the Sixth Book of Euclid, have attained some facility in all ordinary algebraical methods as far as the Binomial Theorem, and have made a beginning with Trigonometry.

The Examples in the first course are therefore of a very easy kind, depending merely on the simpler processes of Arithmetic. On the other hand the chief object of the higher course of questions is to reinforce ordinary lessons in Geometry and Algebra (as well as in Arithmetic) by a series of concrete illustrations. Thus the purpose of the book is not technical. It is designed first of all to supplement the staple subjects of an elementary mathematical training: and for this reason every opportunity has been

used of illustrating the principles of Euclid's Third, Fourth, and Sixth Books by such groups of questions as are found on pages 65, 82, 93 and elsewhere. At the same time it is hoped that this treatment of the subject will be useful as an introduction to those students who may afterwards have occasion to study practical applications of Mensuration in works of a more technical character.

Proofs of formulæ have been given or indicated whenever they seemed likely to be intelligible to learners whose acquirements fall within the limits above mentioned; but it has not been thought well to trench upon the ground of theoretical Solid Geometry. Nor have I considered it within the scope of a first text-book of Mensuration to deal with the numerous applications of the "Prismoidal Formula" and its important practical developments.

The Examples in the body of the book are with few exceptions original, having been framed from time to time for the use of my pupils in Clifton College. To candidates reading for admission to the Royal Military Colleges the set of Miscellaneous Examples to be worked by means of Logarithms (Chapter XXIII.) will, it is believed, be found a useful exercise. In the specimen examples worked out in this chapter I have been glad to follow the arrangement of logarithmic work adopted by my colleague Mr H. S. Hall and the late Dr Knight in their *Elementary Trigonometry*—particularly in the now general practice of discarding the *ten* from the tabular logarithm of a trigonometrical ratio before employing it in calculation. To Mr Hall also I am indebted for some of the questions to be solved by logarithms.

The formulæ of Elementary Mensuration are collected for reference and revision in Chapter XXIV, and are followed

by an exercise consisting of questions selected from recent examination papers.

I have to thank my colleague Mr M. C. Playne for kindly verifying a number of examples in Part II.

F. H. STEVENS.

CLIFTON COLLEGE,
November, 1895.

NOTE

ON THE WEIGHT OF A GIVEN VOLUME OF WATER AND THE CAPACITY OF THE GALLON.

IN the examples of Part II the usually accepted standard (fixed by Act of Parliament until the year 1878) has been retained. This gives the weight of 1 cubic inch of distilled water as 252·458 grains, and the weight of 1 cubic foot of water as 997·137 oz. Av.; and by consequence it determines the capacity of one gallon (10 lbs. Av. of distilled water) as 277·274 cubic inches.

These results vary to some extent according to the methods and instruments used. The latest determination (which has not yet—1895—received legal sanction) gives 252·286 grains as the weight of one cubic inch of distilled water at the temperature 62° Fahrenheit, barometer at 30 inches. Under this determination a cubic foot of water weighs 996·458 oz. Av., and 1 gallon contains 277·463 cubic inches.

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PART I.

PART II.

PART I.

THE MENSURATION OF PLANE FIGURES.

CHAPTER I.

INTRODUCTORY.

1. THE object of Mensuration is to find the lengths of lines, the area of surfaces and the volume of solid figures by means of given rules and formulæ.

2. It is not necessarily within the scope of an elementary text-book on Mensuration to *prove* the rules and formulæ by which it works: these are derived from Geometry, Trigonometry, and other branches of theoretical mathematics. Mensuration teaches their use and practical application.

3. In stating and applying the rules of Mensuration we shall assume that the beginner has a knowledge of so much Algebra as will enable him to understand the meaning of a simple formula. In addition to this he is expected to have read the First Book of Euclid. The methods by which he will work will depend (in the elementary course) on the ordinary rules of Arithmetic, especially the rules of decimals;

and in particular he is strongly advised to acquire at the outset the practice of *Contracted Multiplication* and *Division*, which he should invariably use where approximate results are required.

4. These methods we will explain by examples.

CONTRACTED MULTIPLICATION.

Example i. Multiply 3·6804569 by 25·14651, the result being required true to three places of decimals.

Arrange the decimals one under the other so that the decimal points may be in a vertical column. As the result is to be *true to three places*, it is necessary to retain *four* decimal figures at each stage of the work.

Draw a vertical line after the third decimal figure. We shall begin the multiplication with the *left*-hand figure of the multiplier, that is with the 2, and work from left to right, but before doing so, we erase all decimal figures of the multiplicand beyond the fifth. For since to multiply by 2 in the tens' place is in reality to multiply by 20, the result of each figure must be written one place to the *left* of its natural position, the decimal point being kept in the same column as those above it. It will be seen then, as only four decimal figures are to be retained in the work, that we require (in this case) only five decimal figures in the multiplicand.

```
 3·680|4569
25·146|51
------------
73·609|1
18·402|2
  ·368|0
  ·147|2
  ·022|1
  ·001|8
------------
92·550|
```

It should be noted, however, that in beginning to multiply at the fifth decimal figure of the multiplicand, we must carry to the result the number which would have been carried if the remaining figures 69 had not been erased; thus the right-hand figure in the first line of the work is 1 (not 0).

Now cross out in the multiplicand the next figure 5 on the right, and multiply through by the next figure 5 of the multiplier, this time (as the multiplying figure is in the units' place) writing each result under the figure from which it is derived.

Again, cross out the next figure on the right in the multiplicand, and multiply through by the next figure 1 of the multiplier, ranging the result one place to the right. Continue this process until all the figures in the multiplicand are crossed out, or all the figures in the multiplier are exhausted. Then add, retaining three decimal places in the product.

Example ii. Multiply 106·30168465 by ·028565483, retaining a result true to four places of decimals.

106·3016	8465
·0285	65483
2·1260	3
·8504	1
·0531	5
·0063	8
·0005	3
·0000	4
3·0365	4

NOTE (i) At each stage of the work *five* figures are retained. The first multiplying figure 2 being in the hundredths' place, the result is written two places to the right, thus *three* decimal figures only are retained in the multiplicand.

(ii) In carrying to the first figure in the second line 8 is counted as 10; in the fourth line 18 is counted as 20.

(iii) Had the fifth decimal figure in the product exceeded 5, in rejecting it we should have added *one* to the fourth decimal figure.

EXAMPLES. I. A.

ON CONTRACTED MULTIPLICATION OF DECIMALS.

1. Multiply 3·2508204 by 4·556281, giving your result true to three decimal places.

2. Find the square of 3·14159 to three places of decimals.

3. Multiply 304·06085016 by ·000853421, giving a result true to four decimal places.

4. Multiply £21·68245 by ·13406, and give your result true to the nearest penny.

5. Find to the nearest penny the value of 6·030415 of £7. 8*s*. 2¾*d*.

CONTRACTED DIVISION OF DECIMALS.

6. Definition. The figures of a number or decimal, other than 0's standing at the beginning or end, are called **significant figures**.

For example in 27080900, 2708·09, and ·000270809, the significant figures in each case are 2, 7, 0, 8, 0, 9.

Example i. It is required to divide 79·43896245 by 3·2064025, the quotient to be correct to two decimal places.

First determine by inspection how many *integral* figures there will be in the quotient. In this case there will clearly be *two* integral figures. And as two decimal figures are required, it follows that we have to find the first *four* significant figures of the quotient.

Retain in the divisor *five* figures (including the integer); that is to say, retain one more than the number of significant figures required in the quotient: and in the dividend retain as many figures as are needed to take the first step in the division, in this case, *five*.

Omitting the decimal points, the work will now stand thus:

```
32064|79438|2477
     |64128|
      15310
      12825
       2485
       2244
        241
        224
```

We now proceed with the division in the ordinary way, except that at each stage, instead of bringing down a new figure from the dividend, *we cross off a figure from the right of the divisor*, taking care, however, on multiplying to make use of the figure last crossed off for the purpose of obtaining a carrying number.

Thus we obtain as the quotient the figures 2477: but since it has been already determined that there will be *two* integral figures, the required result is 24·77.

Example ii. Divide ·02628947597 by 3·0685, the result to be correct to *eight* decimal places.

```
3068500|26289475|856753
       |24548000|
         1741475
         1534250
          207225
          184110
           23115
           21478
            1636
            1534
             102
              92
```

Here we see at once (by moving forward the decimal point the same number of places in divisor and dividend) that there will be *two* 0's before the first significant figure in the quotient. Hence to make up the required eight places of decimals we have to find *six* significant figures.

This makes it necessary to retain *seven* figures in the divisor, which is done in this case by adding two 0's. It will be noticed that the first step in the division requires eight significant figures in the dividend.

The decimal points as before are omitted in the numerical work.

Thus the required result is ·00856753.

EXAMPLES. I. B.

On Contracted Division.

1. Divide 286·34895 by 5·3608166, giving the result true to two places of decimals.

2. Divide ·0009477824 by ·04609508, giving the result correct to five places of decimals.

3. Find the first seven decimal figures in the quotient, when ·019824826 is divided by 43·06.

4. Find correct to five decimal places the value of

$$1 \div 3{\cdot}14159.$$

5. Find correct to four decimal places the value of

$$\frac{236{\cdot}405 \times {\cdot}0026054}{4{\cdot}6082}$$

Tables.

6. The following Tables are those chiefly used in Plane Mensuration.

I. Length of Lines, or Linear Measure.

12 inches = 1 foot,
3 feet = 1 yard,
$5\frac{1}{2}$ yards = 1 pole,
40 poles or 220 yards = 1 furlong,
8 furlongs or 1760 yards = 1 mile.

II. The Area of Surfaces, or Square Measure.

144 square inches = 1 square foot,
9 square feet = 1 square yard,
$30\frac{1}{4}$ square yards = 1 square pole,
40 square poles = 1 rood,
4 roods or 4840 square yards = 1 acre.

In the practice of Land-Surveying, which is the most important application of plane mensuration, distances are measured by a *chain* (known as **Gunter's Chain**). This instrument is 22 *yards* in length, and consists of 100 *links*. The length of each link is therefore a little less than 8 inches (7·92 inches).

Thus 22 yards = 1 chain,

∴ $(22)^2$, or 484, square yards = 1 square chain.

But 4840 square yards = 1 acre.

∴ 10 square chains = 1 acre.

Similarly, since 100 links = 1 chain,

∴ $(100)^2$, or 10000, square links = 1 square chain.

∴ 100000 square links = 1 acre.

Thus, square chains are converted into acres by dividing by 10, *or moving the decimal point* one place *to the left.*

For example. 342·5 sq. chains = 34·25, or $34\frac{1}{4}$, acres.

And square links are converted into acres by dividing by 100000, *or moving the decimal point* five places *to the left.*

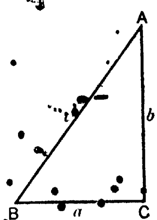
For example, 698245 square links = 6·98245 acres.

CHAPTER II.

THE SIDES OF A RIGHT-ANGLED TRIANGLE.

7. THE object of this chapter is to shew how to find the length of a side of a right-angled triangle, when the lengths of the other two sides are known.

If ABC is a triangle right-angled at C, *the square on the hypotenuse* AB *is equal to the sum of the squares on* BC, CA. [Euc. I. 47.]



Or, $AB^2 = BC^2 + CA^2$.

Thus if BC, CA, AB contain a, b, and c units of length respectively, then

$$c^2 = a^2 + b^2;$$

$$\therefore\ c = \sqrt{a^2 + b^2}.$$

8. *Given the lengths of the two sides including the right angle, to find the length of the hypotenuse.*

Example i. In a right-angled triangle the lengths of the sides including the right angle are respectively 11 yards and 18 yds. 2 ft. Find the length of the hypotenuse.

Here $\left.\begin{aligned} a &= 11 \text{ yards} &&= 33 \text{ feet} \\ b &= 18 \text{ yds. } 2 \text{ ft.} &&= 56 \text{ feet} \end{aligned}\right\}$ Required c.

And $c^2 = a^2 + b^2 = (33)^2 + (56)^2$

$$= 1089 + 3136 = 4225.$$

$$\therefore\ c = \sqrt{4225} = 65 \text{ feet.}$$

$$= \underline{21 \text{ yds. } 2 \text{ ft.}}$$

Example ii. The two sides containing the right angle measure 2 chains 31 links and 1 chain 60 links respectively: find the length of the hypotenuse.

Here $a = 2$ chains 31 links $= 2\cdot31$ chains, $b = 1$ chain 60 links $= 1\cdot6$ chains, Required c.

$$c^2 = a^2 + b^2$$
$$= (2\cdot31)^2 + (1\cdot6)^2$$
$$= 5\cdot3361 + 2\cdot56 = 7\cdot8961.$$
$$c = \sqrt{7\cdot8961}$$
$$= 2\cdot81 \text{ chains} = 2 \text{ chains } 81 \text{ links.}$$

9. *In a right-angled triangle, given the lengths of the hypotenuse and of one side, to find the length of the other side.*

With the notation of Art. 7, it has been shewn that

$$c^2 = a^2 + b^2,$$
$$\therefore\ a^2 = c^2 - b^2;$$
$$\therefore\ a = \sqrt{c^2 - b^2}.$$

Example. The hypotenuse of a right-angled triangle is 15 ft. 5 in. and one side is 8 ft. 8 in.; find the other side.

Here $c = 15$ ft. 5 in. $= 185$ inches; $b = 8$ ft. 8 in. $= 104$ inches, Required a.

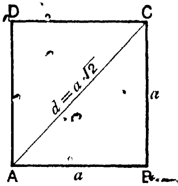
$$a^2 = c^2 - b^2 = (185)^2 - (104)^2$$
$$= 34225 - 10816 = 23409.$$
$$\therefore\ a = \sqrt{23409} = 153 \text{ inches}$$
$$= \underline{12 \text{ ft. } 9 \text{ in.}}$$

NOTE. The numerical work in this example may be abridged by the use of factors.

Thus
$$a^2 = c^2 - b^2 = (c+b)(c-b)$$
$$= (185+104)(185-104)$$
$$= 289 \times 81 = 17^2 \times 9^2$$
$$\therefore\ a = 17 \times 9 = 153.$$

10. *The relation between the side of a square, and its diagonal.*

Let ABCD be a square, of which AC is a diagonal.



Let each side contain a units of length, and the diagonal d units. Then ABC is a right-angled triangle.

$$\therefore AC^2 = AB^2 + BC^2,$$

$$\text{or } d^2 = a^2 + a^2 = 2a^2;$$

$$\therefore d = \sqrt{2a^2} = a\sqrt{2} \quad \ldots\ldots\ldots\text{(i)}.$$

Hence
$$a = \frac{d}{\sqrt{2}} = \frac{d\sqrt{2}}{2} \quad \ldots\ldots\ldots\text{(ii)}.$$

That is,
$$\textit{diagonal} = \textit{side} \times \sqrt{2} \quad \ldots\ldots\ldots\text{(i)},$$

$$\textit{side} = \frac{1}{2}\,\textit{diagonal} \times \sqrt{2} \quad \ldots\ldots\text{(ii)}.$$

NOTE. The value of $\sqrt{2}$ to five places of decimals is 1·41421.

Example i. The side of a square is 22 p. 1 yd. 2 ft.; find to the nearest foot the length of its diagonal.

Here $a = 22$ p. 1 yd. 2 ft. $= 368$ feet.

diagonal $=$ *side* $\times \sqrt{2}$
$= 368 \times 1·41421$ feet
$= 520$ feet, nearly.

1·414	2
368·	
424·2	6
84·8	5
11·3	1
520·4	

Example ii. The diagonal of a square measures 17 chains 32 links; find to the nearest link the length of its side.

Side of square $= \frac{1}{\sqrt{2}}$ *diagonal*
$= \frac{1}{2} \times \sqrt{2} \times 17·32$ chains
$= 1·41421 \times 8·66$ chains
$= 12·25$ nearly
$=$ 12 chains 25 links.

1·41	42
8·66	
11·31	3
·84	8
·08	5
12·24	6

*11. Some questions connected with the sides of right-angled triangle are solved by algebraical equations.

Example. The hypotenuse of a right-angled triangle is 29 feet, and the sum of the two sides containing the right angle is 41 feet. Find the sides.

With the notation of Art. 7 we have

$$c = 29, \text{ and } a+b=41.$$

$$\therefore c^2=841.$$

But $$c^2=a^2+b^2. \quad \text{Euc. I. 47.}$$

$$\therefore a^2+b^2=841 \quad \ldots\ldots\ldots\ldots\ldots(\text{i}),$$

and $$a+b = 41 \quad \ldots\ldots\ldots\ldots\ldots(\text{ii}).$$

Solving the equations (i) and (ii) by the ordinary algebraical method we find

$$\left.\begin{aligned} a&=21, \text{ or } 20.\\ b&=20, \text{ or } 21.\end{aligned}\right\}$$ Result, 21 feet and 20 feet.

EXAMPLES. II. A.

On the Sides of a Right-Angled Triangle.

[*Elementary Course.*]

1. Find the hypotenuse of a right-angled triangle in which the sides containing the right angle measure respectively

(i) 15 feet and 8 feet;

(ii) 35 feet and 12 feet;

(iii) 4 feet and 1 ft. 2 in.;

(iv) 3 yds. 2 ft. and 7 yards;

(v) 4 chains 80 links and 1 chain 40 links;

(vi) 350 links and 120 links;

(vii) 3 p. 2 yds. 0 ft. 6 in. and 2 poles;

(viii) 2 p. 1 yd. and 4 p. 3 yds. 2 ft.

2. Find the remaining side of a right-angled triangle in which the hypotenuse and one of the sides including the right angle are respectively

(i) 29 feet and 20 feet;

(ii) 85 inches and 36 inches;

(iii) 5 ft. 5 in. and 4 ft. 8 in.;

(iv) 32 yds. 1 ft and 24 yds.;

(v) 2 chains 41 links and 1 cha

(vi) 10 chains 90 links and 6 chains;

(vii) 2 p. 5 yds. 2 ft. and 2 p. 5 yds.

3. A ladder whose foot is placed on the ground 36 feet from the front of a house reaches to a window at a height of 48 feet: what is the length of the ladder?

4. If a ship sails 168 miles due North and then 95 miles due East, how far will it be from its starting point?

5. A boat making 12 knots an hour steams for 3 hours due South and 4 hours due West. How far is it then from its starting point?

6. Find to two places of decimals the diagonal of a square whose side is 18 feet.

7. Find to the nearest link the diagonal of a square whose side is 5 chains 11 links.

8. Two ships are observed from a signal-station to bear respectively N.E., 15 miles distant, and N.W., 8 miles distant. How far are they apart?

9. A ladder 29 feet long is placed so as to reach a point in the front of a house 21 feet above the ground: how far is its foot from the house?

10. A balloon at an altitude 1680 feet is 1930 feet distant from the point from which it ascended. How far has it been drifted by the wind?

11. ABC is a triangle, and from A a perpendicular AD is drawn to the base BC.

If AD = 12 inches, BC = 25 inches, and BD = 9 inches; find the lengths of AB and AC.

12. ABC is a triangle, and from A a perpendicular AD is drawn to the base BC.

Given AB = 5 inches, BD = 3 inches, and AC = $6\frac{2}{3}$ inches. Find the lengths of AD and DC.

13. A vessel on leaving port sails due North for 5 hours at the rate of 6 miles an hour, and then due East for 8 hours at the rate of 5 miles an hour. How long would she take to return to port, if she sailed in a direct line at the rate of 10 miles an hour?

14. A vessel making for a port 20 miles distant reaches it in 4 hours by sailing for half this time due East at the rate of 8 miles per hour, and for the rest due North. At what speed must she have sailed Northward?

*EXAMPLES. II. B.

On the Sides of a Right-Angled

[*Higher Course.*]

(*Side and diagonal of a square.*)

1. Find to the nearest foot the diagonal of a square whose side is 6 p. 4 yds. 2 feet.

2. Find in feet correct to two places of decimals the side of a square whose diagonal is 100 feet.

3. A ladder, 40 feet in length, reaches to a window which is as high above the ground as the foot of the ladder is distant from the house. Find the height of the window to the nearest inch.

4. Find to the nearest link the side of a square field whose diagonal is 8 chains 12 links.

5. A man takes 3 minutes to walk round a square enclosure at the rate of 4 miles an hour. Find to the nearest foot the length of the diagonal.

6. A man walks diagonally from corner to corner of a square enclosure in $2\frac{1}{2}$ minutes at the rate of 3 miles an hour: find to the nearest inch the length of a side.

(*Geometrical Questions.*)

7. A man travels 20 miles due North, then 15 miles due East, finally 28 miles due South; what is the distance from his starting point?

8. A man travelled 10 miles due North, a certain distance due East, and finally 31 miles due South. If his distance from home was then 29 miles, how far did he travel due East?

9. ABC is a triangle, and from A a perpendicular AD is drawn to the base BC.

Given AB = 17 feet, AC = 25 feet, and AD = 15 feet. Find BC.

10. ABC is a triangle, and from A a perpendicular AD is drawn to the base BC.

If AB = 37 feet, AC = 20 feet, and AD = 12 feet; find BC.

11. In the quadrilateral ABCD, the angles ABC, ADC are right angles; also AB = 12 feet, BC = 5 feet, and CD = 11·2 feet. Find AD.

12. A ladder 50 feet long is placed so as to reach a window 48 feet high, and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street.

13. A ladder 85 feet long is placed so as to reach a window 77 feet high. Find how high the ladder would reach on being turned over to the opposite side of the street, the breadth of the street being 113 feet.

14. A man walking 4 miles an hour travels 12 miles due North, 10 miles due East, and again 12 miles due North. If he returns to his starting point in a direct line, how long will the whole journey take him?

15. If it costs £5. 12*s.* to fence in a square enclosure, find to the nearest penny the cost of running a fence from corner to corner (diagonally).

16. If a man walk from corner to corner of a square enclosure in $5\frac{1}{2}$ minutes, find to the nearest second how long he would take to walk round it.

(*Questions requiring Algebraical Equations.*)

17. The hypotenuse of a right-angled triangle is 55 feet, and the length of one of the sides containing the right angle is $\frac{3}{4}$ of the other. Find the two sides.

18. The hypotenuse of a right-angled triangle is 52 inches, and one of the sides is $\frac{5}{12}$ of the other. Find the two sides.

19. In a right-angled triangle one of the sides containing the right angle is 6 inches, and the hypotenuse is double of the other side. Find the length of this side in inches to two places of decimals.

20. In a right-angled triangle one of the sides containing the right angle is 12 inches, and the hypotenuse is five times the other side. Find the hypotenuse to the nearest hundredth of an inch.

21. The hypotenuse of a right-angled triangle is 13 feet, and the sum of the sides containing the right angle is 17 feet: find the length of these sides.

22. The hypotenuse of a right-angled triangle is 17 feet, and the difference of the sides containing the right angle is 7 feet: find the length of these sides.

CHAPTER III.

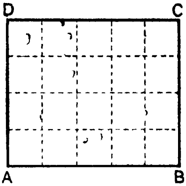
THE AREA OF THE RECTANGLE AND SQUARE.

SECTION I.

12. *If the number of linear units in the length of a rectangle is multiplied by the number of linear units in its breadth, the product gives the number of units of square measure in its area.*

For example, if a rectangle is 5 feet long and 4 feet wide, the area is 5×4 (or 20) square feet.

The reason of this may be thus shewn.



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth BC is 4 feet.

Divide AB into five equal parts, and BC into four equal parts; and through the points of division in each line draw straight lines parallel to the other.

The rectangle ABCD is now divided into a number of compartments, each of which represents the square on a side of one foot, that is, a square foot. Now in the whole rectangle there are *four* rows each containing *five* of these squares; hence the rectangle contains 5×4 (or 20) square feet.

Thus if there are a *linear units in the length of a rectangle and* b *linear units in its breadth, then its area contains* ab *units of square measure.*

And if there are a *linear units in the side of a square, its area contains* $a \times a$, *or* a^2, *units of square measure.*

These statements may be abridged by saying that

the area of a rectangle = *length* × *breadth* ... (i),

the area of a square = $(side)^2$ ....(ii).

Example i. A rectangular area is 5 yds. 1 ft. long and 3 yds. 2 ft. broad: find its area.

Here the length = 5 yds. 1 ft. = 16 feet,

the breadth = 3 yds. 2 ft. = 11 feet;

∴ the area = 16×11 square feet

= 176 square ft.

= 19 sq. yds. 5 sq. ft.

Example ii. The side of a square field measures 6 chains 75 links. Find its area in acres, roods, and poles.

Here each side = 6 chains 75 links = 6·75 chains

$\therefore$ the area $= (6{\cdot}75)^2$ square chains

$= 45{\cdot}5625$ square chains $= 4{\cdot}55625$ acres

$= \underline{4 \text{ ac. } 2 \text{ r. } 9 \text{ p.}}$

Example iii. A rectangular allotment of ground measures 26 chains 16 links in length, and its breadth is five-sixths of its length. Find to the nearest penny what rent should be paid at the rate of £2. 12*s*. 4*d*. an acre.

Here the length = 26 chains 16 links = 2616 links;

the breadth $= \frac{5}{6}$ of 2616 links $= 2180$ links.

$\therefore$ the area $= 2616 \times 2180$ square links

$= 5702880$ square links $= 57{\cdot}0288$ acres.

The rent of 57·0288 acres at £2. 12*s*. 4*d*. an acre.

	£57·0288	
	2	
	114·0576	
10*s*. = £½	28·5144	
2*s*. = $\frac{1}{5}$ of 10*s*.	5·7029	
4*d*. = $\frac{1}{6}$ of 2*s*.	·9505	
	£149·2254	= £149. 4*s*. 6*d*.

EXAMPLES. III. A.

On the Rectangle and Square.

[*Elementary Course.*]

(Given the length and breadth, to find the area.)

1. Find the areas of the following rectangles, giving the results in square feet:—

   (i) Length 3 yds. 2 ft., breadth 1 yard 1 ft.

   (ii) Length 6 yds. 1 ft., breadth 2 yds. 2 ft.

   (iii) Length 5 yds. 2 ft. 3 in., breadth 2 yds. 1 ft. 6 in.

2. Find the areas of the following rectangles, giving the results in acres:—

(i) Length 20 chains, breadth 15 chains.

(ii) Length 25 chains, breadth 5 chains.

(iii) Length 20 chains, breadth 8 chains 25 links.

(iv) Length 1215 links, breadth 1040 links.

(v) Length 99 yards, breadth 4 chains 20 links.

3. Find the areas of the following squares:—

(i) Length of each side 4 ft. 3 in.: give the result in square feet and square inches.

(ii) Length of each side 880 yards: give the result in acres.

(iii) Length of each side 11 chains 30 links: give the result in acres.

(iv) Length of each side 225 links: give the result in roods and poles.

(v) Length of each side 21 chains 75 links; give the result in acres, roods and poles.

4. A rectangular field is 23 chains in length, and it is one-fourth as wide as it is long: what is its acreage?

5. Find the cost of paving a rectangular area, whose length is 40 yards and breadth 33 yards 1 foot, at the rate of 3*d.* per square foot.

6. Find the cost of painting a ceiling 20 ft. 3 in. long by 18 ft. 4 in. wide, at the rate of 1*s.* 6*d.* per square yard.

7. A rectangular tennis-ground is 110 yards long and 55 yards wide. Find the expense of sowing it with grass-seed, at the rate of $3\frac{1}{2}$ bushels to the acre, the price of the seed being £1. 1*s.* 4*d.* per bushel.

8. The perimeter of a square is 93 yards 1 foot: find its area in square yards and square feet.

9. The perimeter of a square is 17 chains; find its area in acres, roods and poles.

10. The perimeter of a square field is 100 chains; what rent must be paid for it, at the rate of £2. 10*s.* for one acre?

(*Contracted Multiplication. Approximation.*)

11. Each side of a square measures 23 chains 7 links: find its area in acres, roods, poles and the nearest square yard.

12. Each side of a square measures 13 chains 28 links: find its area in acres, roods, poles and the nearest square yard.

13. The breadth of a rectangular area is five-sevenths of its length. If the length is 41 chains 23 links, find its area in acres, roods, poles, and the nearest square yard.

14. The perimeter of a square field is 7 chains 21 links: find its area in roods, poles, and the nearest square yard.

15. A rectangular allotment of ground is 39 chains 20 links in length, and its width is three-quarters of its length. Find to the nearest penny what rent should be paid for it, at the rate of £1 an acre.

16. Each side of a square plot of ground measures 21 chains 12 links. Find its area to the nearest square yard; also find (to the nearest penny) the rent at 10*s.* 6*d.* per acre.

SECTION II.

13. *Given the area of a rectangle and the length of one side, to find the other side.*

It has been shewn in the last section that

the area of a rectangle = the length × the breadth.

It follows that $\textit{the length} = \dfrac{\textit{the area}}{\textit{the breadth}}$;

and similarly, $\textit{the breadth} = \dfrac{\textit{the area}}{\textit{the length}}$:

but in applying these formulæ care must be taken that the linear units and the units of area correspond. Thus an area expressed in *square feet* must be divided by the breadth expressed in *linear feet*, the length so obtained being in *linear feet.*

Example i. A rectangular area contains 222 sq. yds. 8 sq. ft., and its breadth is 11 yards 1 ft.: find its length.

Here the area $= 222$ sq. yds. 8 sq. ft. $= 222\frac{8}{9}$ sq. yds.,

the breadth $= 11$ yds. 1 ft. $= 11\frac{1}{3}$ yards;

$\therefore$ the length $= 222\frac{8}{9} \div 11\frac{1}{3}$ yards $= 19\frac{2}{3}$ yards

$= \underline{19 \text{ yds. } 2 \text{ ft.}}$

Example ii. A rectangular area contains 4 ac. 2 r. 9 p., and its length is 12 chains 15 links; find its breadth.

$$\begin{array}{r|l} 40 & 9{\cdot}0 \text{ poles} \\ \hline 4 & 2{\cdot}225 \text{ roods} \\ \hline & \end{array}$$

$\therefore$ 4 ac. 2 r. 9 p. $= 4{\cdot}55625$ acres $= 45{\cdot}5625$ square chains

$$\therefore \text{ breadth} = \frac{45{\cdot}5625}{12{\cdot}15} \text{ chains} = 3{\cdot}75 \text{ chains}$$

$= \underline{3 \text{ chains } 75 \text{ links.}}$

14. *Given the area of a square, to find the length of its side.*

If there are a linear units in the side of a square, it has been shewn that its area contains a^2 units of square measure. Thus if A denote the number of *square* units in the area

$$a^2 = A,$$

$$\therefore a = \sqrt{A}.$$

Hence the number of linear *units in the side is found by taking the square root of the number of* square *units in the area.*

Example. Find to the nearest link the side of a square whose area is 6 ac. 3 r. 4 p. 22 sq. yds.

Express the area in terms of acres and the decimal of an acre.

Thus 6 ac. 3 r. 4 p. 22 sq. yds. $= 6{\cdot}7795454\ldots$ acres

$= 67{\cdot}795454\ldots$ square chains.

Hence the side $= \sqrt{67{\cdot}795454\ldots}$ chains

$= 8{\cdot}233\ldots$ chains

$= \underline{8 \text{ chains } 23 \text{ links}}$ (to the nearest link).

EXAMPLES. III. B.

[Elementary Course.]

(Given the area of a rectangle and the length of one side, to find the other side.

Given the area of a square to find the length of its side.)

1. Find the lengths of the following rectangles, having given that

(i) The area is 195 sq. feet, and the breadth is 13 feet.

(ii) The area is 222 sq. ft., and the breadth is 12 feet.

(iii) The area is 1000 sq. yds., and the width is 75 feet.

(iv) The area is 3524 sq. yds. 8 sq. ft., and the breadth is 51 yds. 1 ft.

2. Find the breadth of the following rectangular fields, having given that

(i) The area is 34 acres, and the length is 20 chains.

(ii) The area is $76\frac{1}{2}$ acres, and the length is 3600 links.

(iii) The acreage is 37·6607, and the length is 23 chains 45 links.

3. Find the sides of the following squares, having given that

(i) The area is 289 sq. ft. (Give the result in feet.)

(ii) The area is 152 sq. yds. 1 sq. ft. (Give the result in yards and feet.)

(iii) The area is $2\frac{1}{2}$ acres. (Give the result in yards.)

(iv) The area is $19\frac{3}{5}$ acres. (Give the result in yards.)

(v) The area is 3610 acres. (Give the result in chains.)

(vi) The area is 656·1 acres. (Give the result in chains.)

(vii) The area is 33·124 acres. (Give the result in links.)

4. The area of a square is 169 sq. ft. What is its perimeter in feet?

5. The area of a square is $22\frac{1}{2}$ acres: find its perimeter in chains.

6. How long will it take a man to walk round a square containing 40 acres, at the rate of four miles an hour?

7. What will it cost to fence in a square ten-acre field at the rate of £2. 12*s.* 3*d.* per 20 yards?

8. The area of a rectangle is equal to the sum of three squares whose sides are 18 feet, 19 feet, 20 feet. If one side of the rectangle is 31 ft., find the other side.

9. A rectangular field is 3 times as long as it is wide, and its area is 7½ acres: find its length and breadth in chains.

10. A rectangular enclosure, 4 times as long as it is wide, contains 119·025 acres: find its length and breadth in chains and links.

11. Find the length of a rectangular figure whose breadth is 49 yards, and whose area is equal to that of a square on a side of 56 yards.

12. Find the length of a rectangular figure whose breadth is 4 yds. 1 ft. and whose area is equal to that of a square on a side of 30 yds. 1 ft.

13. It costs £11. 15*s.* 9*d.* to floor a room with planking at 8*d.* per square foot: if the breadth of the room is 17 ft. 3 in., find the length.

14. The rent of a rectangular plot of ground is £53. 7*s.* 6*d.*, at the rate of 17*s.* 6*d.* per acre: if the length is 40 chains, find the breadth.

15. The rent of a square allotment is £8. 3*s.* 4*d.*, at the rate of 8*s.* 4*d.* per acre; find in yards the length of each side.

16. The annual profits derived from a rectangular plot of ground are £185. 12*s.* 6*d.*, this being at the rate of £2. 15*s.* per acre. If its length is three times its breadth, find both dimensions in chains.

(*Approximation and Contracted Division.*)

17. The area of a rectangular enclosure is 3632·48137 sq. yds. and the breadth is 51·2136 yds.: find the length in yards, feet, and inches, to the nearest inch.

18. A rectangular field has an area of 5 ac. 2 r. 18 p. 26 sq. yds., and its breadth is 7 chains 55 links: find the length to the nearest link.

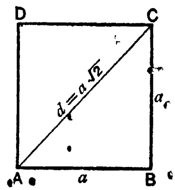
19. A square field contains 14 ac. 3 r. 17 p. 21 sq. yds.: find the length of its side in chains and links, giving the result true to the nearest link.

20. A rectangular plot of ground is valued at £5816. 13*s.* 6*d.*, at the rate of £463. 7*s.* 5*d.* an acre. If the breadth is 6 chains 43 links, find the length to the nearest link.

SECTION III.

MISCELLANEOUS EXAMPLES ON THE RECTANGLE AND SQUARE.

15. ***To find the area of a square whose diagonal is given.***



Let ABCD be a square, of which AC is a diagonal.

Let each side contain a units of length, and the diagonal d units. Then ABC is a right-angled triangle.

$$\therefore AC^2 = AB^2 + BC^2, \quad \text{(Art. 7)}$$

$$\text{or } d^2 = a^2 + a^2 = 2a^2.$$

But the area of the square $= a^2 = \frac{1}{2} \cdot d^2.$

Hence the area of a square is half the square on its diagonal.

Example. The diagonal of a square is 6 chains 50 links; find its area.

Here $d = 6$ chains 50 links $= 6{\cdot}5$ chains.

But the area of square $= \frac{1}{2}d^2 = \frac{1}{2} \times (6{\cdot}5)^2$ square chains

$= 21{\cdot}125$ square chains

$= 2{\cdot}1125$ acres

$= 2$ ac. 0 r. 18 p.

16. Questions connected with the carpeting of rooms are solved by remembering that the area of the floor is equal to the area of the strip of carpet required to cover it.

Example. Find the cost of carpet, 30 inches wide, at 4*s*. 2*d*. a yard, for a room whose length is 20 feet and breadth 18 feet.

The area of the carpet $=$ the area of the floor

$= 20 \times 18$ sq. ft.

The width of the carpet $=$ 30 inches $= 2\frac{1}{2}$ feet.

$\therefore$ the length of the carpet $= 20 \times 18 \div 2\frac{1}{2}$ feet

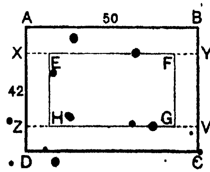
$= 144$ feet $= 48$ yards.

$\therefore$ the cost $= 4s.\ 2d. \times 48 = \underline{£10.}$

17. Another type of question deals with a rectangular space having a second rectangle so placed within it as to leave a margin of uniform width between the two figures.

Example. A court-yard 50 feet long by 42 feet broad contains a rectangular lawn surrounded by a gravel path of uniform width. If the width of the path is 6 feet, find the dimensions and the area of the lawn, and the area of the path.

Let **ABCD** represent the rectangular court-yard and **EFGH** the lawn.



Produce **EF** both ways to **X** and **Y**; and produce **HG** to **Z** and **V**.

Then **XY** $=$ **AB** $= 50$ feet:

but **EX** and **FY** are each 6 feet;

$\therefore$ **EF** $= 38$ feet.

Similarly **EH** $= 30$ feet.

$\therefore$ the area of the lawn

$= 38 \times 30 = 1140$ sq. ft.

Now the path is made up of the figures **AY**, **ZC**, **XH**, **FV**; that is, of twice the figure **AY** with twice the figure **XH**: hence its area is

$2 \times 50 \times 6 + 2 \times 30 \times 6$ square feet $= \underline{960 \text{ sq. ft.}}$

*18. We will now consider a few examples of a somewhat more difficult kind, some of them requiring the solution of algebraical equations.

Example i. The area of a rectangular field is $37\frac{1}{2}$ acres, and its diagonal is $32\frac{1}{2}$ chains. Find the length and breadth.

Let a and b denote the length and breadth in chains.

Then $a^2+b^2=(\text{diagonal})^2=(32\cdot5)^2$ square chains

$=1056\cdot25$ square chains ....(i),

and ab = area $=37\frac{1}{2}$ acres

$=375$ square chains.............................(ii).

Solving (i) and (ii) we have

$$a+b=42\cdot5 \text{ chains};$$

$$a-b=17\cdot5 \text{ chains}.$$

Hence $a=30$ chains, and $b=12$ chains 50 links.

Example ii. A square and a rectangle each have a perimeter of 60 chains, and the difference between their areas is 10 acres. Find the dimensions of the rectangle.

The side of the square is 15 chains; therefore its area is $(15)^2$, or 225 sq. chains.

Let a and b denote the length and breadth of the rectangle in chains.

Then $a+b=30$ (the semi-perimeter)..........(i),

and $225-ab=100$ sq. chains,

or $ab=125$ ....(ii).

Solving (i) and (ii) by the usual method,

we have $a=25$ chains,

$b=5$ chains.

Example iii. A path of uniform width runs round the interior of a court-yard 58 feet long by 48 feet broad, and encloses a lawn whose area is 222 sq. yds. 2 sq. ft. Find the width of the path.

See the figure to Art. 17, p. 23.

Let the path be x feet wide.

Then the dimensions of the lawn are $58-2x$ feet, and $48-2x$ feet.

Hence the area of the lawn is $(58-2x)(48-2x)$ sq. feet.

But by hypothesis the area of the lawn is 222 sq. yds. 2 sq. ft., or 2000 sq. ft.

$$\therefore \ (58-2x)(48-2x)=2000,$$

or $$x^2-53x+196=0.$$

Hence $$x=4 \text{ feet, or } 49 \text{ feet.}$$

Rejecting the last result, since the path cannot be wider than the court-yard, we conclude that the width of the court-yard is 4 feet.

*19. The area of an inclined plane may be found by the method illustrated in the following example.

Example. A sheet of galvanized iron 50 inches wide is placed against the top of a wall 6 feet high, while the lower edge is 5 ft. 5 in. from the foot of the wall: find the area of the sheet of iron.

The length of the sheet is evidently the hypotenuse of a right-angled triangle whose sides are 6 feet and 5 ft. 5 in.

Hence reducing to inches,

$$\text{the length}=\sqrt{(72)^2+(65)^2}=\sqrt{9409}=97 \text{ inches.}$$

$$\therefore \ \text{the area}=97\times 50 \text{ sq. in.}=33 \text{ sq. ft. } 142 \text{ sq. in.}$$

EXAMPLES. III. C.

MISCELLANEOUS EXERCISES ON THE RECTANGLE AND SQUARE.

[*Elementary Course.*]

(*Diagonals and Areas.*)

1. Find in square feet the area of a square whose diagonal is 8 feet.

2. Find in square feet the area of a square whose diagonal is 7 yds. 1 ft.

3. Find in sq. poles, sq. yards, and sq. feet the area of a square whose diagonal is 6 p. 2 yds. 1 ft.

4. The length of a rectangular area is 12 feet, and its diagonal is 15 feet: find its breadth and area.

5. One side of a rectangle is 72 feet, and its diagonal is 97 feet: find the other side and the area.

6. One side of a rectangular area is 20 yards, and its diagonal is 36 yds. 1 ft. Find the cost of paving it at the rate of 4½d. a square foot.

(Perimeters and Areas.)

7. It takes 12 minutes to run round a square enclosure at the rate of 5½ miles an hour: what is its acreage?

8. It costs £339. 7s. 0d. to fence in a square enclosure at £2. 11s. 5d. per 20 yards: what is its acreage?

9. How long will it take a man to walk round a square field containing 62½ acres at the rate of 3 miles an hour?

10. How much will it cost to fence in a square field of 14⅖ acres at the rate of £1. 5s. 1½d. per dozen yards?

11. The perimeter of a rectangle is 56 feet: if the length is 15 feet, find the area.

12. The cost of fencing in a rectangular field is £13. 7s. 0d. at the rate of £1. 2s. 3d. per 20 yards; if the field is 84 yards long, find its width.

(Covering a rectangular area with tiles, turfs, planks, etc.; carpeting rooms.)

13. How many tiles 6 inches long and 5 inches wide will be required to pave a rectangular area which measures 20 feet by 18 feet?

14. A square sheet contains 4800 postage stamps ·8 inches long and ·6 inches wide: what is the length of each side?

15. If 64 planks, 6 ft. 3 in. long and 8¼ in. wide, are required to floor a room 25 feet long; what is its breadth?

16. Find the cost of paving a square room whose side is 20 feet with bricks 9 inches long and 4 inches wide, the bricks being worth 9d. a score.

17. A rectangular plot of ground 50 yards long and 40 yards broad is to be laid with turfs 1½ feet long by 6 inches wide. Find the cost, if the turfs are worth 3 shillings a hundred.

18. How many feet of planking 14 in. wide will be required for the floor of a room 19 ft. 3 in. long and 16 ft. 4 in. wide?

19. How many yards of paper 35 inches wide are needed to cover a wall 22 ft. 6 in. long by 12 ft. 3 in. broad?

20. For a room 21 feet long by 15 feet wide the length of carpet required is 46 yds. 2 ft. What is the width of the carpet in inches?

21. Find the cost of carpeting a room 30 feet long by 21 feet broad with carpet $\frac{1}{2}$ yard wide at 4*s.* 6*d.* a yard.

(Rectangles surrounded by uniform margins.)

22. A court-yard 72 feet long by 56 feet broad contains a rectangular lawn surrounded by a gravel path which is uniformly 7 feet wide: find the length and breadth of the lawn.

23. A room is 22 ft. 6 in. long, by 18 ft. 3 in. wide. What must be the area of a carpet, if when placed it leaves a uniform margin of floor 2 ft. 6 in. wide?

24. A court-yard 80 feet long and 64 feet broad contains a rectangular lawn whose length is 58 feet surrounded by a path of uniform width. Find the width of the path and the breadth of the lawn.

25. A carpet when placed in a room 30 feet long and 24 ft. 6 in. wide, leaves a uniform margin of floor uncovered. If the length of the carpet is 26 ft. 4 in., find the width of the margin and breadth of the carpet.

26. Find in square yards the area of a path 6 feet wide surrounding a lawn whose length is 30 yards and breadth 24 yards.

27. Find in square yards and square feet the area of a path 4 feet wide surrounding a lawn 24 yds. 2 ft. long and 22 yds. 1 ft. broad.

28. A court-yard whose length is 55 yds. 1 ft. and breadth 33 yds. 1 ft. contains a rectangular lawn surrounded by a gravel path 8 feet wide. Find the area of the lawn and of the path in square yards and feet.

29. A carpet is to be provided for a room 24 ft. 4 in. long and 17 ft. 6 in. wide so as to leave a uniform margin 2 feet wide. How many yards of carpet 30 inches wide will be required? And what will it cost to stain the margin at the rate of 6$\frac{3}{4}$*d.* per square yard?

*EXAMPLES. III. D.

MISCELLANEOUS EXERCISES ON THE RECTANGLE AND SQUARE.

[*Higher Course.*]

1. The diagonal of a square field is 14 chains 22 links: find its area in acres, roods, poles and the nearest square yard.

2. One side of a rectangular enclosure is 13 chains 18 links, and the diagonal is 19 chains 32 links: find the area to the nearest pole.

3. If it takes a man a minutes to walk round a square field at the rate of b miles an hour, what is the area of the field in acres?

4. How many sovereigns will it cost to fence in a square field containing a acres, at the rate of b shillings for c yards?

5. If a planks, b feet long and c inches wide, are required for the floor of a room d yards long; what is its breadth in feet?

6. It costs £l to carpet a room a feet long and b feet broad, with carpet at s shillings a yard. What is the width of the carpet in inches?

7. It costs £l to carpet a room a feet long with carpet c inches wide at s shillings a yard: find the breadth of the room in feet.

(*Questions leading to Algebraical Equations.*)

8. The perimeter of a rectangle is 50 feet and its area is 150 square feet: find its length and breadth.

9. The perimeter of a rectangular field is 56 chains, and its area is $19\frac{1}{5}$ acres: find its length and breadth.

10. The area of a rectangle is 120 sq. feet, and its diagonal is 17 feet: find its length and breadth.

11. The area of a rectangular field is 6 acres, and its diagonal is 13 chains: find its length and breadth in chains.

12. The area of a rectangular field is $16\frac{4}{5}$ acres, and its diagonal is 25 chains: how long would it take to run round it at the rate of 6 miles an hour?

13. A square and a rectangle each have a perimeter of 80 feet. If the difference between the areas of the two figures is 100 sq. feet, find the dimensions of the rectangle.

14. A square and a rectangle each have a perimeter of 180 chains. If the difference between the areas of the two figures is $2\frac{1}{2}$ acres, find the dimensions of the rectangle in chains.

15. A court-yard 50 feet long and 35 feet broad contains a lawn whose area is 1000 square feet, surrounded by a path of uniform width. Find the width of the path.

16. A room 23 feet long by 22 feet wide is furnished with a carpet which leaves a uniform margin of flooring all round. If 56 yards of carpet 30 inches wide were used, what is the width of the margin?

17. A path of uniform width runs round the interior of a quadrangle 48 ft. 6 in. long by 36 feet broad, and encloses a lawn whose area is 126 yards. Find the cost of gravel for the path at the rate of 9*d.* a square yard.

(On inclined planes.)

18. A plank 15 inches wide is placed against the top of a wall 8 feet high, while the other end rests on the ground 6 feet from the wall. Find the area of the plank.

19. A green-house covers a square area of 25 sq. yards, and has a lean-to roof inclined to the horizontal at an angle of 60°. Find the area of the roof, and the number of panes of glass, each 12 inches by 7 inches, required for it, allowing 30 sq. feet for wood-work.

20. A green-house covers a square area of 28 sq. yds. 4 sq. ft. and has a lean-to roof inclined to the horizontal at an angle of 45°. Find the area of the roof to the nearest square foot.

21. A square sheet of galvanized iron, each side being 18 feet, rests against a wall, and is inclined to the horizontal at an angle of 60°: what area of ground will it protect from a vertical rain?

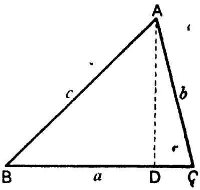
CHAPTER IV.

ON TRIANGLES.

20. In a triangle ABC it is usual to adopt the letters a, b, c to denote the lengths of the sides opposite to the vertices A, B, C.

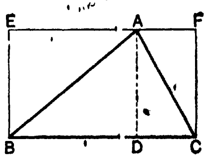
Thus we suppose that BC contains a units of length, CA b units, and AB c units.

The perpendicular AD, drawn from A to the opposite side BC, is called the **altitude** of the triangle relative to BC as base.



21. *To find the area of a triangle having given one side and the perpendicular drawn to it from the opposite vertex.*

If ABC is a triangle whose altitude is AD, and if EBCF is a rectangle on the same base BC and of equal altitude, then by Euclid I. 41



the area of triangle ABC

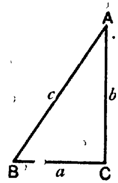
$$= \frac{1}{2} \textit{ area of rectangle } \text{EBCF}$$

$$= \frac{1}{2} \text{BC} \times \text{EB} = \frac{1}{2} \text{BC} \times \text{AD}.$$

That is, $\quad$ *the area of a triangle* $= \frac{1}{2}$ *base* $\times$ *altitude*...(i),

$$\therefore \textit{ the altitude of a triangle} = \frac{\textit{twice the area}}{\textit{base}} \;\ldots\ldots\; \text{(ii)}.$$

22. If the triangle ABC is *right-angled at* C, then the side AC is evidently the *altitude* relative to the base BC.



Hence the area of a right-angled triangle is obtained by taking half the product of the two sides forming the right-angle.

Or, with the previous notation,

$$area = \frac{1}{2} ab.$$

Example i. Find the area of a triangle whose base is 16 chains 15 links and whose altitude is 2 chains 50 links.

$$Area = \frac{1}{2}\ base \times altitude$$
$$= \frac{1}{2} \times 16 \cdot 15 \times 2 \cdot 5 \text{ square chains}$$
$$= 20 \cdot 1875 \text{ square chains}$$
$$= 2 \cdot 01875 \text{ acres} = \underline{2 \text{ ac. } 0 \text{ r. } 3 \text{ p.}}$$

Example ii. The area of a triangle is 4 ac. 0 r. 2 p. and its base is 3 chains 21 links. Find the altitude.

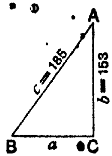
$$4 \text{ ac. } 0 \text{ r. } 2 \text{ p.} = 4 \cdot 0125 \text{ acres} = 40 \cdot 125 \text{ sq. chains}.$$

Now $$altitude = \frac{twice\ area}{base}$$
$$= \frac{80 \cdot 25}{3 \cdot 21} \text{ chains} = \underline{25 \text{ chains.}}$$

Example iii. Find the area of a *right-angled* triangle in which the hypotenuse and one side forming the right angle are respectively 185 feet and 153 feet.

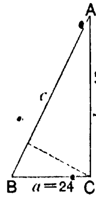
Here $c = 185$ ft., and $b = 153$ ft.: required a.

$$a^2 = c^2 - b^2 = (185)^2 - (153)^2$$
$$= (185 + 153)(185 - 153)$$
$$= 338 \times 32 = 169 \times 64.$$
$$\therefore a = 13 \times 8 = 104 \text{ feet.}$$



But $$area = \frac{1}{2} ab = \frac{1}{2} \times 104 \times 153 \text{ sq. ft.} = \underline{7956 \text{ sq. ft.}}$$

Example iv. In a right-angled triangle, the sides forming the right angle are 24 feet and 45 feet; find the length of the perpendicular drawn from the right angle to the hypotenuse.



Here $a = 24$ ft., $b = 45$ ft.

$$c^2 = a^2 + b^2 = (24)^2 + (45)^2 = 2601.$$

$$\therefore c = 51 \text{ ft.}$$

$$\text{Area} = \frac{1}{2} ab = \frac{1}{2} \times 24 \times 45 \text{ sq. ft.}$$

And *perpendicular on* $AB = \dfrac{\textit{twice area}}{AB}$

$$= \frac{24 \times 45}{51} \text{ feet} = \underline{21 \cdot 2 \text{ feet}}, \text{ nearly.}$$

26. *Given the three sides of a triangle, to find the area.*

The sum of the three sides of a triangle is called its *perimeter.*

The letter s is used to denote the *semi*-perimeter; so that $2s$ denotes the perimeter; thus we have

$$2s = a + b + c$$

and
$$s = \frac{1}{2}(a + b + c).$$

The area of a triangle is denoted by the symbol Δ.

We learn from Trigonometry that *the area of a triangle in terms of its three sides is given by the formula*

$$\Delta = \sqrt{s \,.\, (s-a)(s-b)(s-c)}.$$

Example i. Find the area of a triangle whose sides are 26, 28, and 30 inches.

Here $a = 26$, $b = 28$, $c = 30$;

$$\therefore 2s = 84 \qquad \therefore s = 42$$

$$\therefore \begin{cases} s - a = 42 - 26 = 16 \\ s - b = 42 - 28 = 14 \\ s - c = 42 - 30 = 12 \end{cases}$$

$$\therefore \Delta = \sqrt{42 \times 16 \times 14 \times 12} = \sqrt{112896} = \underline{336} \text{ sq. inches.}$$

NOTE. The labour of multiplication and of finding the square root of the product may often be wholly or in part avoided by breaking up each of the four numbers into its factors, and so re-arranging them as to exhibit the square root.

For instance, $\Delta = \sqrt{42 \times 16 \times 14 \times 12} = \sqrt{7^2 \times 12^2 \times 4^2}$

$= 7 \times 12 \times 4$ sq. in.

$= 336$ sq. in.

24. *Given the three sides of a triangle, to find the perpendicular from any vertex on the opposite side.*

For example, in the triangle ABC, given the values of a, b, and c, to find the perpendicular drawn from A on BC.

First find the area of the triangle by the formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

and then apply the formula

$$altitude = \frac{twice\ area}{base}.$$

Example. In the triangle ABC, $a = 15$ chains 40 links, $b = 10$ chains 90 links, $c = 8$ chains 70 links. Find the perpendicular drawn from A on BC.

Working in links, $a = 1540$,
$b = 1090$,
$c = 870$,
$s = 1750$.

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{1750 \times 210 \times 660 \times 880}$

$= \sqrt{5^2 \times 7^2 \times 3^2 \times 11^2 \times 4^2 \times 100^2}$

$= 462000$ sq. links $= 46 \cdot 2$ sq. chains;

and *perpendicular on* BC $= \frac{twice\ area}{BC} = \frac{92 \cdot 4}{15 \cdot 4}$ chains

$= \underline{6 \text{ chains}}.$

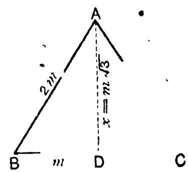
25. *To find the area of an equilateral triangle, and the length of the perpendicular drawn from any vertex to the opposite side.*

For this purpose we might use the formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

and proceed by the method of Art. 24; but it is more instructive in this case to obtain what is required direct from a figure.

Let ABC be an equilateral triangle, and AD the perpendicular drawn from A on BC.



Then D is the middle point of BC (Euc. I. 26).

Suppose the side of the equilateral triangle to contain $2m$ units of length; then BD contains m units. Required AD.

$$AD^2 = AB^2 - BD^2 \qquad \text{(Art. 9)}$$
$$= 4m^2 - m^2$$
$$= 3m^2;$$
$$\therefore AD = m\sqrt{3} \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

And the area of $\triangle ABC = \frac{1}{2} \cdot BC \times AD$

$$= \frac{1}{2} \cdot 2m \cdot m\sqrt{3}$$
$$= m^2\sqrt{3} \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

NOTE. $\sqrt{3} = 1{\cdot}73205\ldots$

Example. Find to the nearest pole the area of an equilateral triangle on a side of 20 chains.

Here $2m = 20$ chains; $\therefore m = 10$ chains.

Now $\Delta = m^2\sqrt{3} = 100 \times 1{\cdot}73205$ sq. chains

$= 17{\cdot}3205$ acres

$= 17$ ac. 1 r. 11 p. nearly.

***26.** The *area of a triangle in terms of two sides and the included angle* is proved in Trigonometry to be given by any one of the expressions

$$\Delta = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B.$$

That is to say *the area is equal to half the product of any two sides multiplied by the* sine *of the included angle.*

Example. A straight line AB of unknown length subtends an angle of 30° at a point O. If OA and OB measure respectively 880 links and 125 links, find the area of the triangle OAB.

Here
$$\Delta = \frac{1}{2} . \mathrm{OA} . \mathrm{OB} \sin \mathrm{AOB}$$
$$= \frac{1}{2} \times 1{\cdot}25 \times 8{\cdot}8 \times \frac{1}{2} \text{ sq. chains}$$
$$= 2{\cdot}75 \text{ sq. chains}$$
$$= \underline{1 \text{ r. } 4 \text{ p.}}$$

***27.** We will now work out an example requiring the aid of Algebraical Equations.

Example. The perimeter of a triangle is 42 inches; if one side is 15 inches, and the area is 84 square inches, find the other two sides.

Here
$$a + b + c = 42, \quad \therefore s = 21,$$
$$a = 15$$
$$\therefore b + c = 27 \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{i}).$$

Also $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 6 \times (21-b)(21-c)} = 84$;

hence
$$21 \times 6 \times (21-b)(21-c) = (84)^2,$$

i.e.
$$(21)^2 - (b+c)\,21 + bc = 56;$$
$$\therefore \text{ from (i), } bc = 182 \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{ii}).$$

Solving (i) and (ii) we have $b = 14$ inches, $c = 13$ inches.

EXAMPLES. IV. A.

On Triangles.

[*Elementary Course.*]

(*Right-angled Triangles.*)

1. Find the areas of the right-angled triangles, in which the two sides including the right angle are

(i) 5 feet and 8 feet; give the result in square feet.

(ii) 7 yds. 1 ft. and 13 yds. [illegible] ft.; give the result in sq. yds. and sq. ft.

(iii) 18 chains 75 links and 16 chains; give the result in acres.

2. Find the areas of the right-angled triangles in which the hypotenuse and one side containing the right angle measure respectively

(i) 37 feet and 12 feet; give the result in square feet.

(ii) 8 yds. 1 ft. and 2 yds. 1 ft.; give the result in sq. yds. and sq. ft.

(iii) 24 chains 10 links and 12 chains; give the result in acres.

3. In a right-angled triangle the area is half an acre and one of the sides containing the right angle is 44 yards; find the other side in yards.

4. In a right-angled triangle, the area is 16 p. 22 sq. yds. and one of the sides containing the right angle is 4 p. 1 yd.; find the other side.

5. Find the hypotenuse of a right-angled triangle in which

(i) The area is 60 sq. ft. and one side containing the right angle is 8 feet.

(ii) The area is $2\frac{2}{5}$ acres, and one side containing the right angle is 8 chains.

(*Base and altitude.*)

6. Find the area of the following triangles in which

(i) The base is 32 feet, and the altitude 17 feet; give the result in square feet.

(ii) The base is 4 yds. 2 ft., and the altitude 3 yds. 1 ft.; give the result in sq. yds. and sq. ft.

(iii) The base is 6 p. 3 yds., and the altitude is 12 p. 5 yds.; give the result in roods, poles, square yards.

(iv) The base is 17 chains 50 links, and the altitude 6 chains 84 links; give the result in acres.

7. Find the altitude and area of the *isosceles* triangles in which

(i) The base is 20 feet, and each of the equal sides 26 feet.

(ii) The base is 6 chains 60 links, and each of the equal sides 6 chains 50 links.

8. Find the altitude of the following triangles, having given that

(i) The area is 56 sq. ft., and the base is 16 feet.

(ii) The area is 60·9 acres, and the base 14 chains 50 links.

9. The area of a triangle is 12 p. 11 sq. yds., and its altitude is 12 p. 2 yds.: find its base.

10. In the following right-angled triangles, find the length of the perpendicular drawn from the right angle to the hypotenuse, when

(i) The sides forming the right angle measure 4 ft. and 3 ft.; give the result in feet.

(ii) The sides forming the right angle measure 24 chains and 7 chains; give the result in chains and links.

(iii) The hypotenuse is 29 feet, and one side forming the right angle is 7 yards; give the result in feet true to one place of decimals.

(*Given the three sides.*)

11. Find the areas of the triangles in which the three sides are respectively

(i) 21 feet, 20 feet, 13 feet; give the result in square feet.

(ii) 21 feet, 17 feet, 10 feet; give the result in square feet.

12. Find the areas of the triangles in which the three sides are respectively

(i) 17 yds., 12 yds. 1 ft.; 6 yds. 2 ft.; give the result in square yards.

(ii) 8 yds. 2 ft., 8 yds. 1 ft., 5 yds. 2 ft.; give the result in square yards and square feet.

(iii) 25 chains, 17 chains, 12 chains; give the result in acres.

(iv) 125 chains, 85 chains, 60 chains; give the result in acres.

(v) 150 links, 130 links, 140 links; give the result as a decimal of an acre.

13. In the triangle ABC, if $a=14$ ft., $b=15$ ft., $c=13$ ft.; find the length of the perpendicular from A on BC.

14. In the triangle ABC, if $a=29$ yds., $b=36$ yds., $c=25$ yds.; find the length of the perpendicular from B on AC.

15. In the triangle ABC, if $a=125$ yds., $b=37$ yds., $c=132$ yds.; find the length of the perpendicular from C on AB.

16. Find the areas of the following equilateral triangles:

(i) Each side is 10 ft.; give the result in square feet true to one decimal place.

(ii) Each side is 18 feet; give the result in square feet true to one decimal place.

(iii) Each side is 25 chains; give the result in acres true to two decimal places.

(*Isosceles Triangles.*)

17. The perimeter of an isosceles triangle is 32 feet, and the base is 12 feet; find the area.

18. The perimeter of an isosceles triangle is 50 feet, and the base is 16 feet; find the area.

19. The area of an isosceles triangle is 168 square feet, and its base is 14 feet; find its equal sides.

20. The area of an isosceles triangle is 60 square feet, and its base is 10 feet; find its equal sides.

(*Miscellaneous.*)

21. ABC is a triangular field right-angled at C. If AC measures 12 chains and BC 5 chains, find

(i) the rent of the field at £5. 6*s.* 8*d.* an acre;

(ii) the cost of planting a hedge from A to B at the rate of half-a-crown a yard;

(iii) the shortest distance, in yards, from C to AB.

22. ABC is a triangular field right-angled at C, whose rent, at the rate of £3. 10*s.* an acre, is £29. 8*s.* If AC measures 24 chains, find the lengths of the other sides in chains, and the shortest distance from C to AB in yards and the decimal of a yard.

*23. ABC is a triangular enclosure, and its sides AB, BC, CA, measure respectively 20 chains, 37 chains, and 51 chains. Find the rent at £4. 3*s.* 4*d.* an acre, and the perpendicular distance (in chains and links) from C to AB.

24. The sides of a triangular field are 25 chains, 17 chains, 28 chains; and its rent is £70. At what rate is this per acre?

*EXAMPLES. IV. B.

ON TRIANGLES.

[*Higher Course.*]

1. What decimal of an acre is the triangle whose sides are 281 links, 231 links, 160 links?

2. What decimal of a square mile is the area of a triangle whose sides are 154 chains, 109 chains and 87 chains?

3. Find in acres, roods, poles and the nearest square yard, the area of a triangle whose sides are

(i) 14 chains, 9 chains, 7 chains.

(ii) 11 chains, 10 chains, 9 chains.

4. The sides of a triangular field are 21 chains, 16 chains, and 11 chains. Find its rent to the nearest penny, at the rate of £1. 2*s.* 8*d.* per acre.

5. In a triangle ABC the sides BC, CA, AB measure respectively 18 yards, 13 yards, and 11 yards. Find, to the nearest inch, the length of the perpendicular from C on AB.

6. The base of an isosceles triangle is 154 feet, and each of *its equal sides is 85 feet. Find, to the nearest inch, the side of a square of equal area.*

7. The sides of a *triangle* are *68 feet, 75 feet, and 77 feet.* Find, to the nearest inch, the side of a square of equal area.

(Equilateral Triangles.)

$$\sqrt{3} = 1 \cdot 73205.$$

8. The perpendicular drawn from a vertex of an equilateral triangle to the opposite side is 7 chains. Find the perimeter in yards and inches, to the nearest inch.

9. The perpendicular drawn from a vertex of an equilateral triangle to the opposite side measures 12 chains; find the area of the triangle in acres to three decimal places.

10. The area of an equilateral triangle is 30 square yards; find the perimeter to the nearest foot.

11. The side of an equilateral triangle is 20 feet. Find, to the nearest inch, the side of a square of equal area.

12. The sides of a triangle are 5 inches, 7 inches, 8 inches. Find the side of an equilateral triangle of equal area. Give your result in inches correct to two places of decimals.

13. The difference between the areas of a square and equilateral triangle described on the same base is $6\frac{1}{2}$ acres. Find, to the nearest yard, the length of the common base.

14. An isosceles triangle and an equilateral triangle are described on the same base of 16 feet: and the area of the first triangle is double that of the other. Find, to the nearest inch, the length of the equal sides.

15. On the sides of a square ABCD, equilateral triangles AXB, BYC, CZD, DVA, are described outside the square. Prove that the figure XYZV is a square; and if AB measures 20 feet, find its area in square feet (to one place of decimals).

16. The sides of an equilateral triangle ABC are 6 feet in length. In AB a point Z is taken 2 feet from A; in BC a point X 2 feet from B; and in CA a point Y 2 feet from C. Shew that the triangle XYZ is equilateral, and find its area to the nearest square inch.

(Questions to be solved algebraically.)

17. *The sides of a triangle are proportional to 3, 4, and 5. If the perimeter is 84 feet, find the sides and the area.*

18. The sides of a triangle are proportional to the numbers 7, 8, 9. The cost of fencing it in at 7*s.* 6*d.* per dozen yards is £9. 15*s.* Find the sides in yards.

19. The sides of a triangle are proportional to the numbers 13, 20, 21. If the area is 1134 square feet, find the sides in feet.

20. The sides of a triangular allotment are proportional to 13, 12, 5. The taxes amount to £19. 13*s.* 9*d.*, this being at the rate of 5*s.* 3*d.* an acre. Find the sides in chains.

21. The perimeter of a *right-angled* triangle is 24 feet, and the area is 24 square feet; find the sides.

22. The perimeter of a *right-angled* triangle is 40 feet, and its area is 60 square feet; find the sides.

23. The perimeter of a triangle is 48 feet. If one side is 10 feet, and the area 84 square feet, find the two remaining sides.

24. The perimeter of a triangle is 54 feet: if one side is 21 feet, and the area 126 square feet, find the two remaining sides.

(*Area* $= \frac{1}{2}$ ab sin C.)

25. Two sides of a triangle are 12 feet and 15 feet respectively, and include an angle of 18°. Find the area in square feet to two decimal places.

26. Two sides of a triangle measure 12 chains 50 links and 24 chains 50 links and include an angle of 60°: find the side of an equilateral triangle of equal area.

27. From a point O within a triangle ABC, it is found that the three sides subtend equal angles. If OA, OB, OC measure 5, 12, and 20 chains respectively, find the area of the triangle.

CHAPTER V.

QUADRILATERALS.

SECTION I.

THE PARALLELOGRAM.

28. DEFINITIONS. A **parallelogram** is a four-sided figure whose opposite sides are parallel.

A **rhombus** is a parallelogram whose sides are all equal, and whose angles are not right-angles.

Of a parallelogram we know, from Euclid I. 34, that

(i) *the opposite sides are equal;*

(ii) *the opposite angles are equal;*

(iii) *either diagonal divides the figure into two triangles equal in all respects.*

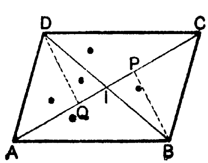
It may also be shewn that

(iv) *the diagonals of a parallelogram bisect one another;*

(v) *the diagonals of a* rhombus *bisect one another at* right angles,

(vi) *the perpendiculars drawn from one pair of opposite vertices on the diagonal joining the other pair, are equal.*

Thus if ABCD is a parallelogram whose diagonals AC, BD intersect at I, and if BP, DQ are perpendiculars drawn from B and D to AC, then AI = IC; BI = ID; and BP = DQ.



The two triangles ABC, ADC are equal in all respects. The four triangles AIB, BIC, CID, DIA are equal in area. [Euc. I. 38.]

29. *To find the area of a parallelogram.*

(i) *Given two adjacent sides and a diagonal.*

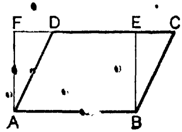
Suppose the lengths of AB, BC, and CA to be known. Then the area of the triangle ABC may be found by the formula $\sqrt{s(s-a)(s-b)(s-c)}$. And the area of the whole parallelogram is double the area of this triangle.

(ii) *Given the diagonal joining two vertices and the perpendicular on it from one of the remaining vertices.*

Suppose the lengths of AC and BP are given.

Then the area of the triangle ABC is found from the formula $\frac{1}{2}$ *base* × *altitude*, or $\frac{1}{2}$ AC × BP: hence the area of the whole parallelogram is AC × BP

(iii) *Given the base and height.*



Let ABCD be a parallelogram and ABEF a rectangle on the same base AB and of the same height BE: then

area of par^{m}. ABCD

= area of rect. ABEF [Euc. I. 35]

= AB × BE.

Hence the *area of a parallelogram* = *base* × *height*.

Example i. Find the area of a parallelogram ABCD, having given that AB = 26 feet, BC = 28 feet, and the diagonal AC = 30 feet.

Here the triangle ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{42 \times 16 \times 14 \times 12} = 336$ sq. feet.

∴ the area of the whole parallelogram = 672 sq. feet.

Example ii. Find the acreage of a parallelogram ABCD, having given that the diagonal AC is 17 chains 50 links and the perpendicular from B on AC is 6 chains 84 links.

Here the area = AC × BP = 17·5 × 6·84 sq. chains

= 119·7 sq. chains = 11·97 acres.

4. The area of a parallelogram is 17·5 acres, and each of two parallel sides measures 21 chains: find to the nearest link the perpendicular distance between them.

5. Find the area of a rhombus on a side of 10 inches, one diagonal being also 10 inches.

6. The diagonals of a rhombus are 4 feet and 1 ft. 2 in.; find the sides and the area.

[*Higher Course.*]

7. Two adjacent sides of a parallelogram are 231 ft. and 120 ft. and the perpendicular distance between the pair of shorter sides is 77 feet: find the distance between the other pair.

8. In a parallelogram the perpendiculars between the two pairs of parallel sides are 65 ft. and 91 ft. If one side is 119 feet, find the adjacent side.

9. A field is in the form of a rhombus whose diagonals are 2870 links and 1850 links; find to the nearest penny the rent at £4. 10*s*. 6*d*. an acre.

10. The diagonals of a parallelogram are 34 feet and 24 feet, and one side is 25 feet. find its area.

11. Find the area of a parallelogram of which one side is 15 chains 40 links in length, and the diagonals are 21 chains 80 links, and 17 chains 40 links respectively.

12. A rhombus is drawn on a side of 1 ft. 8 in., one angle of the figure being 75°. Find its area, and the perpendicular distance between a pair of parallel sides.

13. Find the area of a parallelogram of which two adjacent sides measure 2 ft. 6 in. and 3 ft. 4 in. and are inclined at an angle of 52° 26′. Find also the perpendicular distance between its longer sides. [$\sin 52° 26' = \cdot 7926445$.]

14. The area of a rhombus on a side of 18 inches is 162 sq. in. Find the angles of the figure, and the perpendicular distance between a pair of parallel sides.

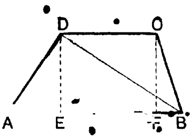
15. The area of a parallelogram is 17·32 acres; if the diagonals measure respectively 25 chains and 16 chains, at what angle are they inclined to one another? [$\sqrt{3} = 1 \cdot 732$.]

SECTION II.

THE TRAPEZIUM.

33. DEFINITION. A **trapezium** is a four-sided figure having *one* pair of opposite sides parallel.

34. *To find the area of a trapezium.*



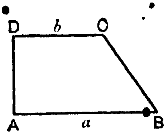
Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.

Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

$$\begin{aligned}\text{Then the area of ABCD} &= \triangle ABD + \triangle DBC\\ &= \frac{1}{2}AB \cdot DE + \frac{1}{2}DC \cdot CF\\ &= \frac{1}{2}ah + \frac{1}{2}bh\\ &= \frac{h}{2}(a+b).\end{aligned}$$

That is, *the area of a trapezium* $= \frac{1}{2}$ *height* × (*the sum of the parallel sides*).

35. The special case when two adjacent angles are right angles is of much importance.



In the adjoining figure the angles at A and D are right angles, so that AD is the *height*. Hence we have

$$\text{the area} = \frac{1}{2} \cdot AD\,(a+b).$$

Example. The parallel sides of a trapezium are 2344 links and 1156 links respectively; and the distance between them is 6 chains 80 links. Find the area.

Here $a = 2344$, $b = 1156$, and $h = 680$ links.

$$\text{Area} = \frac{h}{2}\{a+b\} = 340 \times 3500 \text{ sq. links}$$
$$= 1190000 \text{ sq. links}$$
$$= 11 \cdot 9 \text{ acres}$$
$$= 11 \text{ ac. } 3 \text{ r. } 24 \text{ p.}$$

EXAMPLES. V. B.

THE TRAPEZIUM.

[*Elementary Course.*]

1. Find the area of a trapezium when

(i) The two parallel sides are 6 ft. 2 in. and 7 ft. 4 in., and the perpendicular distance between them is 4 feet.

(ii) The two parallel sides are 7 yds. 1 ft. 5 in. and 4 yds. 2 ft. 7 in., and the perpendicular distance between them is 6 yards.

2. ABCD is a quadrilateral having the sides AD and BC parallel, and the angle ABC a right angle. If BC, AD and AB measure respectively 2883 links, 2117 links and 1624 links, find the area in acres, roods and poles.

3. ABCD is a quadrilateral having the sides AD and BC parallel, and the angle ABC a right angle. If BC, AD and AB measure respectively 32 chains 11 links, 29 chains 14 links, and 24 chains 50 links: find (in yards) the side of a square which has one-fifth the area.

4. The area of a trapezium is 7 acres, and the two parallel sides are 8 chains 25 links, and 5 chains 75 links: find the perpendicular distance between them.

5. If land costs £1500 an acre, find the cost of a quadrilateral field which has two parallel sides measuring 1169 links and 851 links, the perpendicular distance between them being 350 links.

[*Higher Course.*]

6. A quadrilateral field has two parallel sides measuring 1346 links and 1154 links, the perpendicular distance between them being 620 links. If the rent is £38. 15*s.*, at what rate is this per acre?

7. A quadrilateral field has two parallel sides measuring 17 chains 37 links and 15 chains 13 links. If the rent at £2 an acre is £55. 18*s.*, find the distance between the two parallel sides.

8. The two parallel sides of a trapezium measure 58 yards and 42 yards respectively; the other sides are equal, each being 17 yards. Find the area.

9. The two parallel sides of a trapezium measure 13 chains 60 links, and 6 chains 40 links; the other sides are equal, each being 8 chains 50 links. Find the area.

10. In the trapezium ABCD the angles at A and D are right angles, and the angle BCD is 120°. If DC = 40 feet, CB = 20 feet, find the area in square feet to two decimal places.

SECTION III.

ANY QUADRILATERAL.

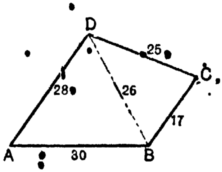
36. The area of an irregular quadrilateral is found by drawing a diagonal, and calculating the areas of the two triangles thus formed. Hence any data from which we can find the areas of the two triangles, will enable us to determine the area of the quadrilateral.

Example i. ABCD is a quadrilateral in which the following measurements have been taken:

AB = 30 in., BC = 17 in.,

CD = 25 in., DA = 28 in.,

and the diagonal BD = 26 in.



Here the area of each of the triangles ABD, BCD is to be found by the formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

The $\triangle ABD = \sqrt{42 \times 12 \times 14 \times 16} = \sqrt{7^2 \times 6^2 \times 8^2} = 336$ sq. in.

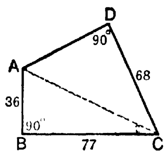
The $\triangle BCD = \sqrt{34 \times 8 \times 9 \times 17} = \sqrt{17^2 \times 4^2 \times 3^2} = 204$ sq. in.

$\therefore$ the area of the quadrilateral $= 336 + 204$ sq. in.

$= 540$ sq. in.

Example ii. ABCD is a quadrilateral in which the angles ABC, CDA are right angles: also AB = 36 chains, BC = 77 chains, and CD = 68 chains. Find the area.

Here we have to find the area of the two *right-angled* triangles ABC, ADC.



First find the length of AD

Now $AC^2 = AB^2 + BC^2 = 36^2 + 77^2$

$= 7225.$

$\therefore AC = 85.$

And $AD^2 = AC^2 - CD^2 = 85^2 - 68^2$

$= (85 + 68)(85 - 68)$

$= 17^2 \times 3^2.$

$\therefore AD = 51.$

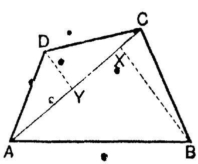
Now the area of $\triangle ABC = \frac{1}{2} AB . BC = 1386$ sq. chains.

And the area of $\triangle ADC = \frac{1}{2} AD . DC = 1734$ sq. chains.

$\therefore$ the area of the quad^l $= 3120$ sq. chains

$= 312$ acres.

37. But the most convenient way of finding the area of a quadrilateral is by means of the diagonal joining one pair of vertices, and the perpendiculars drawn to it from the other pair. Such perpendiculars are called **offsets**. Thus in the adjoining figure BX and DY are said to be the offsets from AC to B and D.



If AC contains d units of length, and BX, DY respectively p and q units,

$$\text{the area of the quad}^{l}.\ \text{ABCD} = \triangle\text{ABC} + \triangle\text{ADC}$$
$$= \frac{1}{2}\text{AC}.\text{BX} + \frac{1}{2}\text{AC}.\text{DY}$$
$$= \frac{1}{2}dp + \frac{1}{2}dq$$
$$= \frac{1}{2}d(p+q)$$

that is to say,

the area of a quadrilateral $= \frac{1}{2}$ *diagonal* $\times$ (*sum of offsets*).

Example. Find the area of a quadrilateral ABCD in which the diagonal AC measures 5 chains 25 links, and the perpendiculars on it from B and D are 3 chains 72 links and 4 chains 28 links respectively.

Here $d = 5 \cdot 25$ chains, $p = 3 \cdot 72$ chains, $q = 4 \cdot 28$ chains.

$$\textit{Area} = \frac{1}{2}d(p+q) = \frac{1}{2} \times 5 \cdot 25 \times 8 \text{ sq. chains}$$
$$= 21 \text{ sq. chains} = 2 \cdot 1 \text{ acres}$$
$$= 2 \text{ ac. } 0 \text{ r. } 16 \text{ p.}$$

COR. 1. If the quadrilateral is such that its *diagonals* (d, d') *are at right angles to one another*, the area is given by the formula

$$\textit{area} = \frac{1}{2}dd'.$$

For by reference to the figure it will be seen that in this case the sum of the offsets from the diagonal d is the other diagonal d'.

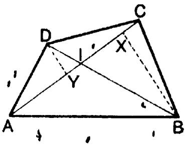
Hence $$\textit{area} = \frac{1}{2}d(p+q) = \frac{1}{2}dd'.$$

*COR. 2. If the diagonals are inclined to one another at an angle I°, *the area is* $\frac{1}{2} dd' \sin I$.

For with the previous notation,

$p = BX = BI \sin I$,

$q = DY = DI \sin I$.



But area $= \frac{1}{2} d (p + q)$

$= \frac{1}{2} d (BI + DI) \sin I$

$= \frac{1}{2} dd' \sin I$.

EXAMPLES. V. C.

ANY QUADRILATERALS.

[*Elementary and Higher Course.*]

(*Given a diagonal and offsets.*)

1. ABCD is a quadrilateral figure in which the diagonal AC measures 18 inches, and the perpendiculars on it from B and D are 11 inches and 9 inches; find the area in square inches.

2. In the quadrilateral ABCD, the diagonal AC measures 850 links, and the perpendiculars on it from B and D are 513 links and 487 links respectively. Find the acreage.

3. In the quadrilateral ABCD, the diagonal AC measures 4250 links, and the offsets from it to B and D are 1763 links and 1117 links respectively: find the area in acres, roods and poles.

4. Find the rent at £4. 4*s.* an acre, of a quadrilateral plot of ground of which the line joining two opposite vertices measures 15 chains, and the offsets from it to the other vertices are respectively 11 chains 81 links and 8 chains 44 links.

(Given the sides and a diagonal.)

5. Find (in square inches) the area of the quadrilateral ABCD in which AB = 13 inches, BC = 20 inches, CD = 17 inches, DA = 10 inches, and the diagonal AC = 21 inches.

6. In the quadrilateral ABCD the following measurements have been made: AB = 26 chains, BC = 17 chains, CD = 17 chains, DA = 12 chains, AC = 25 chains. Find the area in acres, roods and poles.

(Perpendicular diagonals.)

7. The diagonals of the quadrilateral ABCD are perpendicular to one another, and measure 5 yds. 1 ft. and 2 ft. 3 in. respectively. Find the area in square yards.

8. In the quadrilateral ABCD it is observed that the diagonals AC and BD are perpendicular to one another, and measure respectively 1625 links and 2480 links. Find the area in acres, roods and poles.

(Sides and angles.)

9. Find (in square feet) the area of the quadrilateral ABCD, in which the angles ABC and CDA are right angles, and AB = 15 feet, BC = 20 feet, CD = 7 feet.

10. Find the area of the quadrilateral ABCD, in which the following measurements have been ascertained. AB = 35 feet, BC = 12 feet, CD = 20 feet, DA = 51 feet, and the angle ABC is a right angle.

11. The following measurements have been taken of a quadrilateral field ABCD. AB = 40 yards, BC = 75 yards, CD = 77 yards, and the angles ABC, ADC are right angles. What would be the cost of turfing the field at the rate of 2s. per dozen square yards?

12. Find the area (in acres, roods, poles, and the nearest square yard) of the quadrilateral figure ABCD, given that the angle ABC is a right angle, and that AB = 7 chains 20 links, BC = 2 chains 10 links, AD = 6 chains, DC = 4 chains 50 links.

13*. Find the acreage (correct to two decimal places) of the quadrilateral ABCD; having given that the angle ABC is 60°, the angle ADC is a right angle; AB = 13 chains, BC = 13 chains, and CD = 12 chains.

14*. Find the area (in square feet correct to two decimal places) of the quadrilateral figure ABCD; having given that the angle ABC is a right angle, the angle ADC is 60°, AB = 14 feet, BC = 48 feet, and CD = 50 feet.

15*. The diagonals of a quadrilateral are perpendicular to one another, and their lengths are in the ratio 3 : 5. If the area of the figure is $\frac{3}{4}$ acre, find the length of each diagonal.

16*. The two diagonals of a quadrilateral figure are inclined to one another at an angle of 45°, and measure 16 chains and 21 chains 25 links respectively. Find the acreage correct to two decimal places.

17*. The area of a quadrilateral is one acre and the two diagonals measure 8 chains and 5 chains respectively. At what angle are the diagonals inclined to one another?

CHAPTER VI

THE CIRCLE.

SECTION I.

THE CIRCUMFERENCE AND AREA.

38. If the circumference of *any* circle were measured, its length would be found to be nearly $\frac{22}{7}$ of the length of its diameter.

The real value of the ratio which the circumference of a circle bears to its diameter, is *incommensurable*; that is to say, it cannot be *exactly* expressed in figures, though it may be found with any required degree of accuracy. For example, to seven decimal places its value is 3·1415926....

The ratio of the circumference of a circle to its diameter is denoted in mathematics by the Greek letter π. Thus we have

$$\frac{circumference}{diameter} = \pi$$

$$\therefore\ circumference = \pi \times diameter.$$

Or, if c denotes the length of the circumference, and r the radius, then $2r$ denotes its diameter,

and $$\frac{c}{2r} = \pi,$$

so that $$c = 2\pi r;$$

where to π we may give the values

$$\frac{22}{7},\ 3{\cdot}1416,\ \text{or}\ 3{\cdot}1415926,$$

according to the degree of accuracy required in the result.

Example i. Find the circumference of a circle whose radius is 4 yds. 1 ft. 5 in. $\left(\pi = \frac{22}{7}.\right)$

Here $r = 4$ yds. 1 ft. 5 in. $= 161$ inches,

and circumference $= 2\pi r = 2 \times \frac{22}{7} \times 161 = 1012$ inches

$= 28$ yds. 0 ft. 4 in.

Example ii. The driving wheel of a locomotive engine, 6 ft. 3 in. in diameter, makes 110 revolutions a minute. Find the rate at which it is travelling. $(\pi = 3{\cdot}1416.)$

Circumference of wheel $=$ diameter $\times \pi$

$= 6{\cdot}25 \times 3{\cdot}1416$ feet

$= 19{\cdot}6350.$

$$\begin{array}{r|l} 3{\cdot}1416 & \\ 6{\cdot}25 & \\ \hline 18{\cdot}8496 & \\ {\cdot}6283 & 2 \\ {\cdot}1570 & 8 \\ \hline 19{\cdot}6350 & \end{array}$$

Hence in *one minute* the engine travels $19{\cdot}635 \times 110$ feet,

$\therefore$ in *one hour* .... $19{\cdot}635 \times 110 \times 60$ feet

$= 24{\cdot}54...$ miles.

Example iii. If the driving-wheel of a bicycle makes 560 revolutions in travelling a mile, what is its radius? $\left(\pi = \frac{22}{7}.\right)$

The circumference of the wheel $= \frac{1760 \times 3}{560}$ feet $= \frac{66}{7}$ feet.

$$\therefore \quad 2\pi r = \frac{66}{7}.$$

$$\therefore \quad r = \frac{66}{7} \div 2\pi = \frac{66}{7} \times \frac{7}{44} \text{ feet}$$

$$= 1 \text{ ft. } 6 \text{ in.}$$

39. *The area of a circle is found by multiplying the square of the radius by* π.

That is, $\qquad area = \pi r^2.$

Example i. Find the area of a circle whose radius is 8 ft. 9 in. $\left(\pi = \frac{22}{7}.\right)$

Here $\qquad r = 8 \text{ ft. } 9 \text{ in.} = 8\frac{3}{4} \text{ ft.}$

$$\text{area} = \pi r^2 = \frac{22}{7} \times (8\tfrac{3}{4})^2 \text{ sq. ft.}$$

$$= \frac{22}{7} \times \frac{35}{4} \times \frac{35}{4} \text{ sq. ft.}$$

$$= 240\tfrac{5}{8} \text{ sq. ft.} = 240 \text{ sq. ft. } 90 \text{ sq. in.}$$

Example ii. Find to the nearest square yard the area of a circle whose diameter is 42 chains 10 links. $(\pi = 3{\cdot}1416.)$

Here $r = 21{\cdot}05$ chains.

$\text{area} = \pi r^2$

$= 3{\cdot}1416 \times (21{\cdot}05)^2$ sq. chains

$= 1392{\cdot}0508$ sq. chains

$= 139{\cdot}20508$ acres

$= 139$ ac. 0 r. 32 p. 24·6 sq. yds.

21·05	
21·05	
421·0	
21·05	
1·0525	
443·1025	
3·1416	
1329·3075	
44·3102	5
17·7241	0
·4431	6
·2658	6
1392·0508	

40. *Given the area of a circle to find the radius.*

Since $$\pi r^2 = \text{area},$$

$$\therefore\ r^2 = \frac{\text{area}}{\pi};$$

$$\therefore\ r = \sqrt{\frac{\text{area}}{\pi}}.$$

Hence to find the radius, divide the area by π and take the square root of the result.

41. The connection between the area and the circumference may be illustrated by the following example.

Example. The area of a circle is $96\frac{1}{4}$ acres, find its circumference.

Here the area being given, the radius must first be found.

$$\pi r^2 = 96{\cdot}25 \text{ acres} = 962{\cdot}5 \text{ sq. chains}.$$

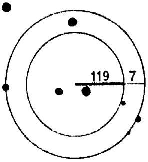
$$\therefore\ r^2 = 962{\cdot}5 \times \frac{7}{22} = 306{\cdot}25 \text{ sq. chains}.$$

$$\therefore\ r = \sqrt{306{\cdot}25} = 17{\cdot}5 \text{ chains}.$$

Now circumference $= 2\pi r = 2 \times \frac{22}{7} \times 17{\cdot}5$ chains $= 2420$ yards.

42. The area of a *plane circular ring* enclosed between two concentric circles is found by taking the difference between the areas of the two circles.

Example i. Find the area of a gravel walk 7 feet wide surrounding a circular pond whose diameter is 238 feet. $\left(\pi = \frac{22}{7}.\right)$



The radius of the inner circle is 119 ft.

The radius of the outer circle is 126 ft.

The area of the outer circle $= \pi\,(126)^2$ sq. ft.

The area of the inner circle $= \pi\,(119)^2$ sq. ft.

$\therefore$ the area of the path

$$= \pi\,(126)^2 - \pi\,(119)^2 \text{ sq. ft.}$$

$$= \pi\,\{(126)^2 - (119)^2\} \text{ sq. ft.}$$

$$= \pi\,(126 + 119)\,(126 - 119) \text{ sq. ft.}$$

$$= \frac{22}{7} \times 245 \times 7 \text{ sq. ft.} = 5390 \text{ sq. ft.}$$

**Example ii.* The internal radius of a plane circular ring is 114 feet, and its area is 7480 sq. feet. Find the width of the ring.

Let the width of the ring be x feet.

Then the external radius is $114 + x$ feet.

Area of outer circle $= \pi (114 + x)^2$ sq. feet.

Area of inner circle $= \pi (114)^2$ sq. feet.

$$\therefore \quad \pi\{(114 + x)^2 - (114)^2\} = \text{area of ring.}$$
$$= 7480 \text{ sq. feet.}$$

$$\therefore (114 + x)^2 - (114)^2 = 7480 \times \frac{7}{22} = 2380 \text{ sq. ft.,}$$

or, $$(114 + x)^2 = 12996 + 2380 = 15376 \text{ sq. ft.}$$

Taking the square root,

$$114 + x = 124 \text{ feet}$$
$$\therefore \quad x = 10 \text{ feet.}$$

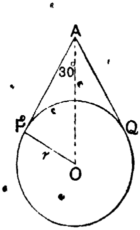
*43. The area of a circle can be found if we know the length of two tangents drawn from an external point, and the angle which they make with one another.

In solving such questions it must be remembered that

(i) the tangents drawn from an external point to a circle are equal; Euc. III. 17. Cor.

(ii) a line drawn from the centre to the point of contact is perpendicular to the tangent. Euc. III. 18.

Example. Two tangents drawn from an external point to a circle are 21 inches in length, and make an angle of 60° with one another. Find the area of the circle.



In the adjoining figure AP = 21 inches, and the angle PAQ = 60° $\therefore$ the angle PAO = 30°.

And PO = PA tan 30° $= \frac{21}{\sqrt{3}}$ inches.

Area of circle $= \pi r^2 = \frac{22}{7} \cdot \left(\frac{21}{\sqrt{3}}\right)^2$ sq. inches

$$= \frac{22}{7} \times \frac{21 \times 21}{3} = 462 \text{ sq. inches.}$$

EXAMPLES. VI. A.

THE CIRCUMFERENCE AND AREA OF A CIRCLE.

[*Elementary Course.*]

(*Circumferences.*)

Take $\pi = \frac{22}{7}$.

1. Find the length of the circumference of the circles whose radii are respectively

(i)	7 inches.	(v)	1 yd. 1 ft. 1 in.
(ii)	$3\frac{1}{2}$ inches	(vi)	1 p. 5 yds.
(iii)	1 ft. 9 in.	(vii)	7 chains 28 links.
(iv)	4 yds. 2 ft.	(viii)	1 chain 5 links.

2. Find the radii of the circles whose circumferences are respectively

(i)	264 inches.	(iii)	1 fur. 24 poles.
(ii)	73 yds. 1 ft.	(iv)	7 chains 92 links.

3. A wire may be bent into the form of a circle of radius 35 inches. If the same wire were bent into the form of a square, what would be the length of its side?

4. A wire may be so bent as to enclose a square whose area is 121 square inches. If the same wire were bent into the form of a circle, what would its radius be?

5. A fountain has a circular basin of radius 21 feet. What would be the cost of enclosing it with iron palings at the rate of 13*s.* 4*d.* a yard?

6. A thin gold wire evenly coiled goes 48 times round a reel. If the value of the wire at 1*s.* 9*d.* a foot is £7. 14*s.*, what is the diameter of the reel?

7. How far has a bicycle travelled, when its driving wheel, 30 inches in diameter, has made 6300 revolutions?

8. A three-mile race is to be run on a circular track whose radius is 84 yards; how many times must the winner run round?

9. How many revolutions are made by the driving wheel of a tricycle, 36 inches in diameter, in travelling 5 miles?

10. The driving wheel of a locomotive engine, 7 feet in diameter, makes 120 revolutions a minute. At what rate is the train travelling?

11. The driving wheel of a locomotive engine, 5 feet in diameter, makes 168 revolutions a minute. At what rate is the train travelling?

(Areas.)

Take $\pi = \frac{22}{7}$.

(Given the radius, to find the area.)

12. Find the areas of the circles whose radii are respectively

(i)	7 inches.	(v)	1 yd. 1 ft. 1 in.
(ii)	$3\frac{1}{2}$ inches.	(vi)	1 p. 5 yds.
(iii)	1 ft. 9 in.	(vii)	3 chains 50 links.
(iv)	4 yds. 2 ft.	(viii)	1 chain 5 links.

13. Find the cost of paving a circular court whose diameter is 42 feet, at the rate of 5*s.* 6*d.* a square yard.

14. Find the value of a circular plot of ground whose diameter is 770 yards, at the rate of £56. 16*s.* an acre.

(Given the area, to find the radius.)

15. Find the radii of the circles whose areas are respectively

(i)	154 sq. in.	(iii)	68 sq. yds. 4 sq. ft.
(ii)	3850 sq. ft.	(iv)	15·4 acres

16. A circular plate of metal costs £1. 13*s.* 0*d.*, this being at the rate of $3\frac{1}{2}$*d.* a square inch; what is the radius of the plate?

17. The rent of a circular plot of ground is £82. 10*s.*, this being at the rate of 10 guineas an acre. Find the radius in chains.

18. Find the radius of a circle equal in area to the sum of two circles whose radii are 4 yards and 11 yards 2 feet.

19. Find the radius of a circle equal in area to the sum of three circles whose radii are 8 inches, 9 inches, 12 inches.

20. The radius of a circle is 5 inches. What is the radius of another circle whose area is nine times that of the first?

(*Area and Circumference.*)

21. Find the area of the circles whose circumferences are

(i) 22 inches.

(ii) 110 feet.

22. Find the circumferences of the circles whose areas are

(i) 616 sq. in.

(ii) 86·625 sq. ft.

(*Circular rings.*)

23. Find the area of a plane circular ring whose external and internal radii are 11 and 4 inches respectively.

24. Find the area of a plane circular ring whose external and internal diameters are 3 ft. 8 in. and 2 ft. 6 in. respectively.

25. A circular lawn 220 yards in diameter is surrounded by a path 12 feet in width. Find the area of the path in square yards.

26. A circular plate of metal, whose diameter is 13 inches, is enclosed in a wooden frame of uniform width. If the width of the frame is one inch, what is its area?

(*Miscellaneous.*)

27. From a square sheet of metal a circular portion is cut. If the side of the square is 9 feet and the radius of the circle 3 ft. 6 in.; find the value of the remainder at the rate of 1*s.* 2*d.* per square foot.

28. From a rectangular garden 88 feet long by 65 feet wide, five circular flower-beds are cut. The central bed has a diameter of 30 feet, and each of the other four a diameter of 20 feet. Find the cost of turfing the rest of the garden at 2*s.* $7\frac{1}{2}$*d.* for 5 square yards.

29. Find in acres, roods, poles and the nearest sq. yd. the area of a circle whose diameter is 8 chains 40 links.

30. Find to the nearest ounce the pressure on a circular plate of metal, whose diameter is 8 inches, at the rate of 16 lbs. per square inch.

EXAMPLES. VI. Γ.

ON THE CIRCUMFERENCE AND AREA OF A CIRCLE.

[*Higher Course.*]

(*Circumferences.*)

Take $\pi = 3{\cdot}1416$.

1. The equatorial diameter of the earth being given as 7925·5 miles, calculate the length of the equator, (i) taking π as $\frac{22}{7}$, (ii) taking π as 3·1416. Find the difference to the nearest mile in the two results.

2. Find in yards, to three decimal places, the radius of a circle whose circumference is equal to the perimeter of a square containing $2\frac{1}{2}$ acres.

3. If the driving wheel of a bicycle, 32 inches in diameter, makes 1000 revolutions in travelling $2792\frac{1}{3}$ yards, calculate the value of π to three decimal places.

4. The earth's orbit is nearly circular. If the mean radius is taken as 92,700,000 miles, find approximately (in miles per second) the velocity with which the earth describes her orbit. [A year $= 365\frac{1}{4}$ days.]

5. The circumference of the larger wheel of a bicycle is to that of the smaller wheel as 45 : 11, and the smaller wheel revolves 272 times more than the other in going a quarter of a mile. Find the circumference of each wheel.

(*Areas.*)

Take $\pi = \frac{22}{7}$.

6. What is the acreage of a circular plot of ground if it cost £247. 10*s.* to fence it in, at the rate of £2. 5*s.* a chain?

7. If it costs £264 to fence in a circular plot of ground at the rate of five guineas a chain, what is the land worth at £101. 10*s.* an acre?

8. How many discs of metal each one inch in diameter may be laid on a plank 20 inches long by 14 inches wide; and how many square inches of plank remain uncovered?

Take $\pi = 3{\cdot}1416$.

9. Find in inches, to two places of decimals, the side of a square whose area is equal to that of a circle of radius 5 inches.

10. Find in chains and links (to the nearest link) the radius of a circle whose area is equal to that of a square on a side of 250 links.

11. A circular disc of metal 20 inches in diameter is beaten into a square plate of the same thickness as the disc. If the side of the plate is found to be 17·724 inches, calculate the value of π to three decimal places.

(*Circular rings.*)

Take $\pi = \frac{22}{7}$.

12. A fountain has a circular basin 31 feet in diameter surrounded by a paved walk 4 feet wide. Find the cost of laying down pavement at the rate of 1*s.* 6*d.* a square yard.

13. The bull's eye of a circular target is 16 inches in diameter, and is surrounded by a red band 5 inches wide. The red band is surrounded by a blue band 4 inches wide. Compare the areas of the red and blue bands.

14. A ring enclosed between two concentric circles has an area of 1320 square inches, and the outer circle has a diameter of 44 inches: find the radius of the inner circle.

15. The external radius of a plane circular ring is 1 foot, and the area of the ring is $34\frac{3}{7}$ square inches: find its width.

16. The internal radius of a plane circular ring is 20 inches, and the area of the ring is 264 square inches: find its width.

17. The external radius of a plane circular ring is 5 yds. 1 ft. and its area is 39 sq. yds. 1 sq. ft.: find the width of the ring.

18. The external diameter of a plane circular ring is 10 yards. and its area is 56 sq. yds. 2 sq. ft.: find the width of the ring.

(Tangents.)

19. From an external point A a tangent AP is drawn to a circle whose centre is O. If AO and AP measure respectively 25 and 24 inches, find the area of the circle.

20. The circumference of a circle of which O is the centre is 220 inches, and A is an external point. If AO is 37 inches, find the length of the tangent drawn from A to the circle.

21. Two tangents drawn from an external point to a circle are 1·75 inches in length and make an angle of 120° with one another. Find the area of the circle.

22. Two tangents drawn from an external point to a circle are at right angles to one another, and measure $1\frac{3}{4}$ in. in length. find the area of the circle.

(Miscellaneous.)

23. From a square sheet of metal whose side is 15 inches nine equal circular pieces are removed: if the area of the remainder is 71 square inches, what is the radius of each of the parts removed?

24. How many circular discs each 8 inches in diameter must be laid on a rectangular board 88 inches by 16 inches to leave an area of 352 square inches uncovered?

25. Two concentric circles are described so that the area of the ring enclosed between them is equal to the area of the smaller circle. Compare the two radii. If the radius of the outer circle is 100 inches, find the radius of the other.

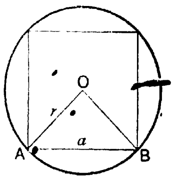
26. Of three concentric circles the radii of the greatest and least are 10 inches and 8 inches respectively. What must be the radius of the middle circle if it divides the ring enclosed between the greatest and the least into two equal parts?

SECTION III.

CIRCLES AND SQUARES: CIRCLES AND TRIANGLES. INSCRIBED AND CIRCUMSCRIBED CIRCLES.

***44.** The following questions are designed to illustrate the principles of Euclid's Fourth Book.

Example i. Find the side of a square inscribed in a circle whose circumference is 132 inches. $\left(\pi=\frac{22}{7}.\right)$



Here $2\pi r = 132$ inches.

$$\therefore\ r = 132 \times \frac{1}{2} \times \frac{7}{22} = 21 \text{ inches}.$$

Then from the right-angled isosceles triangle AOB,

$$\begin{aligned} AB &= r\sqrt{2} = 21 \times 1{\cdot}414 \\ &= 29{\cdot}70 \text{ inches nearly}. \end{aligned}$$

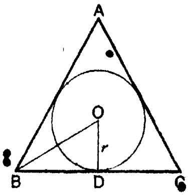
Example ii. The difference between the areas of a circle and its inscribed square is 504 feet. Find the radius of the circle. $\left(\pi=\frac{22}{7}.\right)$

Here the area of circle $=\pi r^2$; the area of square $=(r\sqrt{2})^2=2r^2$.

$$\therefore \left(\frac{22}{7}-2\right) r^2 = 504 \text{ sq. feet},$$

so that $$r = 21 \text{ feet}.$$

Example iii. Find the area of a circle inscribed in an equilateral triangle whose side is 14 inches.



In the adjoining figure BD = 7 inches.

And $r = BD \tan 30° = 7 \cdot \frac{1}{\sqrt{3}}$ inches.

$$\therefore \text{ area of circle} = \pi r^2 = \frac{22}{7} \times 49 \times \frac{1}{3}$$

$$= 51\tfrac{1}{3} \text{ sq. inches}.$$

***45.** The following formulæ give the lengths of the inscribed and circumscribed circles of any triangle.

If a, b, c denote the lengths of the sides, s the semi-perimeter, Δ the area of the triangle, and r, R the *inscribed* and *circumscribed* radii,

then $$r=\frac{\Delta}{s},\ R=\frac{abc}{4\Delta}.$$

*EXAMPLES. VI. D.

CIRCLES AND SQUARES. CIRCLES AND TRIANGLES.

[*Higher Course.*]

Take $\pi=\frac{22}{7}$; $\sqrt{2}=1\cdot41421$

(*Circles and Squares.*)

1. Find (in inches to two places of decimals) the circumference of the circle (i) inscribed in, (ii) circumscribed about a square on a side of 16 inches.

2. Find (in square inches to two places of decimals) the area of the circle (i) inscribed in, (ii) circumscribed about a square on a side of 12 inches.

3. A square field has an area of $2\frac{1}{2}$ acres. Find the circumference of its inscribed circle (in yards), and the area of its circumscribed circle (in acres).

4. Find the side of a square (i) inscribed in, (ii) circumscribed about a circle whose area is 3850 square inches. Give the result in inches.

5. Find the side of a square if the difference between the circumference of its circumscribed and inscribed circles is 44 inches.

6. The difference between the areas of a square and its inscribed circle is 1050 square inches. Find the side of the square.

(Circles and Equilateral Triangles.)

$$\sqrt{3} = 1{\cdot}73205.$$

7. Find the area of the circle (i) inscribed in, (ii) circumscribed about an equilateral triangle whose side is 42 inches.

8. Find the side of an equilateral triangle (i) inscribed in, (ii) circumscribed about a circle of area 462 square inches.

9. A square and an equilateral triangle are drawn on the same base, and the difference between the areas of their inscribed circles is 924 square inches. Find the common base.

10. A square and an equilateral triangle have the same perimeter. Find the ratio of the areas (i) of their inscribed circles, (ii) of their circumscribed circles.

(Circles and Triangles.)

11. In the triangle ABC, right-angled at C, find the radii of the inscribed and circumscribed circles when

(i) $a=15$ inches, $b=8$ inches.

(ii) $a=35$ inches, $b=12$ inches.

12. In the triangle ABC find the values of r and R, having given

(i) $a=21$ inches, $b=20$ inches, $c=13$ inches.

(ii) $a=51$ feet, $b=37$ feet, $c=20$ feet.

13. The area of a right-angled triangle is 84 square inches, and the radius of its circumscribed circle is 12·5 inches: find the sides.

14. In a triangle, whose area is 84 square inches, it is given that $R=8\frac{1}{8}$ inches, $r=4$ inches, $c=15$ inches: find a and b.

15. In the case of the triangle whose sides are 26 feet, 25 feet, and 17 feet, show that the radius of the inscribed circle is equal to $\dfrac{p_1p_2p_3}{p_2p_3+p_3p_1+p_1p_2}$, where p_1, p_2, p_3 are the perpendiculars from the vertices on the opposite sides.

CHAPTER VII.

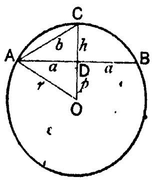
ON CHORDS AND ARCS OF CIRCLES.

SECTION I.

CHORDS.

46. DEFINITIONS. (i) The straight line joining any two points on the circumference of a circle is called a chord of the circle.

(ii) The chord of an *arc* is the straight line joining its extremities.



Let ACB be a circle whose centre is O, and let AB be a chord in it.

From O draw OD perpendicular to AB, and produce OD to meet the circumference at C. Join AC. Then we learn from Euclid, Book III., that

(i) D is the middle point of AB;

(ii) C is the middle point of the arc ACB.

Let the chord AB be $2a$ units in length; so that AD $= a$:

let CD, the height of the arc, $= h$:

let AC, the chord of half the arc, $= b$:

let the radius OA $= r$, and the perpendicular OD $= p$.

If the chord and height of the arc ACB are known, the chord of half the arc is found by Euc. I. 47,

$$b^2 = a^2 + h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

If the chord and height of the arc ACB are known, the radius of the circle is found thus:

$$r^2 = a^2 + p^2 = a^2 + (r - h)^2$$
$$= a^2 + r^2 - 2rh + h^2;$$
$$\therefore\ 2rh = a^2 + h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Combining the results of (i) and (ii) we have

$$b^2 = 2rh \text{.......................} \text{(iii)}.$$

47. In solving the questions which follow, we recommend the student to draw a figure in each case, and work out each example independently, instead of mechanically using the formulæ just proved.

Example i. Find the chord of an arc whose height is 32 inches in a circle of radius 65 inches.

Referring to the last figure we have

$$AO = OC = 65; \text{ and } CD = 32; \therefore OD = 33.$$

Hence $$AD = \sqrt{AO^2 - OD^2} = \sqrt{65^2 - 33^2} = \sqrt{98 \times 32} = 56 \text{ in.}$$

$$\therefore \text{ the chord } AB = 112 \text{ inches.}$$

Example ii. The chord of an arc is 24 inches and its height is 8 inches: find the diameter of the circle.

Let r denote the radius: then we have

$$AD = 12;\ OA = r;\ OD = OC - DC = r - 8.$$

And $$OA^2 = AD^2 + OD^2,$$

or $$r^2 = 144 + (r-8)^2 = 144 + r^2 - 16r + 64.$$

$$\therefore 16r = 208.$$

$$\therefore r = 13.$$

Hence the diameter $= 26$ inches.

Example iii. The height of an arc is 2 inches and the radius is 9 inches: find the chord of half the arc.

Here we may use the formula $b^2 = 2rh$

$$= 2 \times 9 \times 2.$$

$$\therefore b = 6 \text{ inches;}$$

or working direct from the figure, we have

$$OD = OC - DC = 7; \text{ and } OA = 9.$$

$$\therefore AD^2 = 9^2 - 7^2 = 32.$$

But $$AC^2 = AD^2 + DC^2 = 32 + 4 = 36.$$

$\therefore$ AC, the chord of half the arc, $= 6$ inches.

EXAMPLES VII. A.

Chords of a Circle.

[*Elementary and Higher Course.*]

1. In a circle whose radius is 37 inches, a chord is drawn 70 inches in length: find its distance from the centre.

2. In a circle of radius 85 feet there are two parallel chords whose lengths are 72 feet and 102 feet respectively: find their distance apart.

3. Find the chord of an arc whose height is 2 ft. 1 in. in a circle of radius 8 ft. 1 in.

4. In a circle of diameter 4·82 inches, the chord of an arc is 4·18 inches: find the height of the arc.

5. The chord of an arc is 48 inches, and its height is 18 inches: find the radius of the circle.

6. The chord of an arc is 5 yds. 1 ft., and its height is 2 feet: find the diameter of the circle.

7. In a circle of radius 27 inches, the height of an arc is $1\frac{1}{2}$ inches: find the chord of half the arc.

8. In a circle of diameter 4 ft. 2 in., the height of an arc is $4\frac{1}{2}$ inches: find the chord of half the arc.

9. The height of an arc is 7 inches, and the chord of half the arc is 2 ft. 11 in.: find the diameter of the circle.

10. The chord of an arc is 4 feet, and the chord of half the arc is 2 ft. 1 in.: find the radius of the circle to the nearest hundredth of an inch.

11. In a circle of radius 9 inches, the chord of half an arc is 12 inches: find the chord of the whole arc in inches correct to two decimal places.

12. In a circle of diameter 3 ft. 9 in., the chord of half an arc is 1 ft. 3 in.: find the chord of the whole arc in inches correct to two places of decimals.

*13. Find the height of an arc which subtends an angle of 22° 14′ at the centre of a circle of diameter 200 inches.

[Given cos 11° 7′ = ·9812366.]

SECTION II.

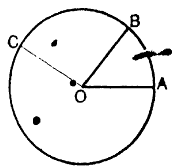
ARCS OF A CIRCLE.

48. DEFINITION. An arc of a circle is a part of the circumference.

If ABC is a circle of which O is the centre, then the angle AOB may be called the **central angle** of the arc AB.

49. *It is proved in* Euclid VI. 33 *that in any circle arcs are proportional to their central angles.*

Thus in the adjoining figure the arc AB : the arc AC :: the $\angle$ AOB : the $\angle$ AOC.



Now the central angle corresponding to the whole circumference is *four right angles*, or 360°: hence, if D denote the number of degrees in the angle AOB, we have

$$\textit{the arc } \mathrm{AB} : \textit{the whole circumference} :: \mathrm{D}^\circ : 360^\circ,$$

$$\text{or} \quad \frac{\textit{arc } \mathrm{AB}}{\textit{circumference}} = \frac{\mathrm{D}}{360}.$$

$$\therefore \ \textit{arc} = \frac{\mathrm{D}}{360} \times \textit{circumference}.$$

Example i. The radius of a circle is 420 feet: find the length of an arc whose central angle is 24° 9′. $\left(\pi = \frac{22}{7}.\right)$

Here $\mathrm{D} = 24^\circ\, 9' = 24{\cdot}15$ degrees.

And
$$\textit{arc} = \frac{\mathrm{D}}{360} \times \textit{circumference}$$
$$= \frac{\mathrm{D}}{360} \times 2\pi r$$
$$= \frac{24{\cdot}15}{360} \times 2 \times \frac{22}{7} \times 420 \text{ feet}$$
$$= \underline{\underline{177{\cdot}1 \text{ feet.}}}$$

Example ii. The radius of a circle is 100 inches: what angle is subtended at the centre by an arc 75 inches in length? ($\pi = 3{\cdot}1416$.)

Let D denote the number of degrees in the required angle: then

$$\frac{D}{360} = \frac{arc}{circumference}$$

$$\therefore\ D = \frac{arc}{2\pi r} \times 360^\circ$$

$$= \frac{75 \times 360}{2 \times 3{\cdot}1416 \times 100}\ \text{degrees}$$

$$= 42{\cdot}9717\ \text{degrees}$$

$$= 42^\circ\ 58'\ 18''\ \text{nearly.}$$

$$\begin{array}{r l} \cdot 9717 & \text{degrees} \\ 60 & \\ \hline 58{\cdot}302 & \text{minutes} \\ 60 & \\ \hline 18{\cdot}12 & \text{seconds} \end{array}$$

Example iii. The length of an arc of a circle is 143 inches, and its central angle is 9° 6′. Find the radius.

Here $\quad D = 9{\cdot}1$ degrees,

and $\quad \dfrac{2\pi r}{\text{arc}} = \dfrac{360}{D}$,

$$\therefore\ r = \frac{360 \times \text{arc}}{D \times 2\pi} = \frac{360 \times 143 \times 7}{9{\cdot}1 \times 2 \times 22}\ \text{inches}$$

$$= 900\ \text{inches.}$$

*50. If θ denotes the circular measure of the central angle (that is, its measure in terms of a *radian*), it is proved in Trigonometry that

$$\theta = \frac{\text{arc}}{\text{radius}},$$

$$\therefore\ arc = r\theta.$$

51. The length of the arc of a circle may also be found *approximately* from the following formula,

$$arc = \frac{8b - 2a}{3},$$

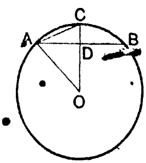
where $2a$ is the chord of the arc, and b the chord of half the arc.

NOTE. This formula gives the length of the arc slightly less than it should be, the error being very small, if the central angle of the arc is small.

If the central angle of the arc is 45°, the formula gives its length to within ·00005 of its real value: and the length of a semi-circumference is given to within ·01 of its true length. Hence if the required arc is greater than a semi-circumference the length of the corresponding *minor* arc must be found, and subtracted from the whole circumference.

Example. In a circle of radius 37 inches find the length of the minor arc whose chord is 24 inches.

In the accompanying figure



$OA = OC = 37$ inches;

$AD = 12$ inches $= a$.

$OD = \sqrt{(37)^2 - (12)^2} = \sqrt{49 \times 25}$

$= 35$ inches.

$\therefore$ $CD = 2$ inches.

$\therefore$ $AC = \sqrt{12^2 + 2^2} = \sqrt{148}$

$= 12{\cdot}165$. inches $= b$.

$\therefore$ $arc = \dfrac{8b - 2a}{3} = \dfrac{97{\cdot}32 - 24}{3} = 24{\cdot}44$ inches (nearly).

EXAMPLES. VII. B.

ARCS OF A CIRCLE.

[*Elementary Course.*]

Take $\pi = \frac{22}{7}$.

1. The radius of a circle is 21 inches: find the length of an arc which subtends an angle of 60° at the centre.

2. The radius of a circle is 9 ft. 4 in.: find the length of an arc which subtends an angle of 11° 15′ at the centre.

3. The radius of a circle is 56 inches: what angle is subtended at the centre by an arc 33 inches in length?

4. The radius of a circle is 45 inches: what angle is subtended at the centre by an arc 39·6 inches in length?

5. The length of an arc of a circle is 55 inches, and it subtends an angle of 30° at the centre; find the radius of the circle.

6. The length of an arc of a circle is 51 yds. 1 ft. and it subtends an angle of 5° 37′ 30″: find the radius of the circle.

[*Higher Course.*]

Take $\pi = 3{\cdot}1416$.

7. The driving wheel of a bicycle 32 inches in diameter has made 35 revolutions. Find to the nearest degree through what angle it must now be turned to complete a journey of 100 yards.

8. A knot (or nautical mile) is the length of *one minute* of arc at the equator. Given that the equatorial diameter of the earth is 7925·5 miles, find to the nearest mile the length of 500 knots.

9. A ship steams due south at the rate of 14 miles an hour. Through how many degrees of latitude will she have passed in three days, given that the earth's mean diameter is 7912 miles?

10. In a circle of radius 10 feet an arc 8 inches in length subtends an angle of 3° 49′ 11″. Calculate the value of π to three decimal places.

$$\left(Arc = \frac{8b - 2a}{3}.\right)$$

11. The chord of an arc is 48 inches, and the chord of half the arc is 27 inches. find approximately the length of the arc.

12. The chord of an arc is 48 inches, and its height is 7 inches: find approximately the length of the arc.

13. In a circle of diameter 40 ft. 2 in., find to the nearest inch the length of the minor arc whose chord is 20 feet.

14. In a circle of radius 32 inches, find to the nearest hundredth of an inch the length of an arc whose height is 9 inches.

15. In a circle of diameter 72 inches, find approximately the length of an arc whose height is 8 inches.

16. Apply the formula to find approximately the length of an arc of 120° in a circle of radius 10 inches; and express as a decimal (correct to the first significant figure) the ratio of the error to the true value.

CHAPTER VIII.

SECTORS AND SEGMENTS OF CIRCLES.

SECTION I.

SECTORS OF CIRCLES.

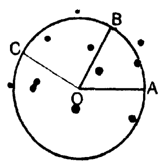
52. DEFINITIONS. (i) A **sector** of a circle is a figure bounded by an arc and the two radii drawn to its extremities.

(ii) The **angle of a sector** is the angle included by the two radii.

53. From Euclid VI. 33, we learn that *the areas of sectors of the same circle are proportional to their angles.*

That is to say, in the figure given below,

the sector AOB : *the sector* AOC :: *the* $\angle$ AOB : *the* $\angle$ AOC.



Now the whole circle may be regarded as a sector whose angle is four right angles, or 360°. Hence if D denotes the number of degrees in the angle AOB, we have

sector AOB : *area of circle* :: D° : 360°,

that is, $\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{D}{360}$,

or $$\text{area of sector} = \frac{D}{360} \times \text{area of circle} \quad \ldots\ldots\ldots\text{(i)}.$$

Similarly we deduce from Euc. VI. 33,

area of sector : *area of circle* :: *arc of sector* : *circumference*,

or $$\frac{\text{area of sector}}{\pi r^2} = \frac{\text{arc}}{2\pi r}.$$

$$\therefore \text{area of sector} = \frac{1}{2}\,\text{arc} \times \text{radius} \ldots\ldots \quad \text{(ii)}.$$

12. The minute-hand of a clock is 5 feet long: in how many minutes will it have swept over an area of 942·48 sq. in.?

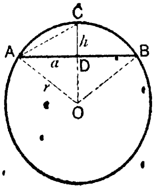
13. Find the area of the greatest sector that can be cut from an equilateral triangle on a side of 10 inches, the centre of the sector being at a vertex of the triangle.

14. Find approximately the area of a sector of a circle of radius 241 inches, the chord of the arc being 240 inches. [Use the formula of Art. 51.]

SECTION II.

SEGMENTS OF CIRCLES.

55. DEFINITION. A **segment** of a circle is the figure bounded by an arc and its chord.



In the accompanying figure O is the centre of the circle of which ACB is an arc. OC bisects AB at right angles, so that C is the middle point of the arc ACB [Euc. III. 30].

Then *area of segment* ACB

= *sector* OACB − *triangle* OAB.

Example. Find the area of a segment cut from a circle of radius 42 inches, if the chord subtends at the centre an angle of 120°.

Area of the sector $= \frac{120}{360}$ *of area of the circle*

$= \frac{1}{3} \times \frac{22}{7} \times (42)^2$ sq. inches = 1848 sq. inches.

And the triangle AOD is half an equilateral triangle; so that since

OD = 21 inches, AD = $21\sqrt{3}$ inches.

∴ *area of the triangle* AOB = $21 \times 21\sqrt{3}$ sq. inches

= 763·8 sq. inches, nearly.

Area of the segment = *sector* − *triangle*

= 1848 − 763·8 sq. inches

= 1084·2 sq. inches.

56. The following examples will illustrate a method of finding approximately the area of a segment,

(i) when the *chord* and *radius* of the circle are given;

(ii) when the *chord* and *height* of the segment are given.

Example i. Find approximately the area of a segment cut off by a chord whose length is 14 inches from a circle of radius 25 inches.

In the figure of the last article we see that

$$OD = \sqrt{25^2 - 7^2} = 24 \text{ inches},$$

$$\therefore \textit{the triangle } AOB = \frac{1}{2} AB \,.\, OD = \frac{1}{2} \times 14 \times 24 = 168 \text{ sq. inches}.$$

The area of the *sector* will be found by the formula

$$\textit{sector} = \frac{1}{2}\,\textit{arc} \times \textit{radius},$$

the length of the arc having first been approximately found by the formula

$$\textit{arc} = \frac{8b - 2a}{3} \quad \text{[Art. 51]}.$$

Now $h = OC - OD = 25 - 24 = 1$ inch,

and $b^2 = AD^2 + DC^2 = 49 + 1 = 50$ inches.

$\therefore\ b = 5\sqrt{2}$ inches $= 7{\cdot}0711$ inches.

Thus
$$\textit{the arc} = \frac{8b - 2a}{3} = \frac{56{\cdot}5688 - 14}{3}$$
$$= 14{\cdot}1896 \text{ inches}.$$

Now
$$\textit{the sector} = \frac{1}{2}\,\textit{arc} \times \textit{radius}$$
$$= \frac{1}{2} \times 14{\cdot}1896 \times 25 \text{ sq. inches}$$
$$= 177{\cdot}37 \text{ sq. inches}.$$

Finally, *the segment* = *sector* − *triangle*
$$= 177{\cdot}37 - 168 \text{ sq. inches}$$
$$= \underline{9{\cdot}37 \text{ sq. inches}}.$$

[NOTE. The area of the segment thus given is *approximate* because the formula for the length of the arc is approximate. For the degree of accuracy see Art. 51. If the required segment is greater than a semicircle, find the area of the conjugate segment, and subtract it from the area of the whole circle.]

Example ii. Find approximately the area of a segment whose chord is 14 inches, and height 1 inch.

Here we must first find the radius.

$$b^2 = 7^2 + 1^2 = 50 \text{ sq. inches.}$$

$$\therefore \; b = 5\sqrt{2} \text{ inches.}$$

And since $b^2 = 2hr$ [Art. 46], $\therefore \; r = 25$.

From this point proceed as in the last Example.

***57.** If θ is the *circular measure* of the angle **AOB**, it will be seen from Arts. 54, 26, that

$$\textit{area of sector} = \frac{1}{2} r^2\theta,$$

$$\textit{area of triangle} = \frac{1}{2} r^2 \sin\theta,$$

$$\therefore \text{ area of segment} = \text{sector} - \text{triangle}$$

$$= \frac{1}{2} r^2\theta - \frac{1}{2} r^2 \sin\theta$$

$$= \frac{1}{2} r^2 (\theta - \sin\theta).$$

EXAMPLES. VIII. B.

THE AREA OF SEGMENTS OF CIRCLES.

[*Elementary and Higher Course.*]

$(\pi = \frac{22}{7}.)$

1. Find the area of a segment of a circle whose chord, 7 inches in length, subtends a right angle at the centre.

2. The chord of a segment is $10\frac{1}{2}$ inches long, and subtends an angle of 60° at the centre. Find the area of the segment.

3. Find the area of a segment cut from a circle of radius 3·5 inches, if the chord subtends an angle of 120° at the centre.

(*Method of Art.* 56.)

4. Find the area of a segment whose chord is 14 inches and radius 25 inches.

5. A segment is cut off from a circle of diameter 482 inches by a chord whose length is 240 inches: find the area of the segment.

6. Find the area of a segment whose chord is 30 inches and height is 8 inches.

7. Find the area of a segment whose chord is 24 inches and height 5 inches.

$$\left(\textit{Area of segment} = \frac{r^2}{2}(\theta - \sin\theta).\right)$$

$$\pi = 3{\cdot}1416.$$

8.* Find the area of a segment of a circle of radius 10 inches, the chord subtending at the centre an angle of 75°.

9.* Find the area of a segment of a circle of radius 8 inches, the chord subtending at the centre an angle of 38° 25′.

[$\sin 38^\circ\, 25' = {\cdot}6214$.]

10.* Find the area of the minor segment cut from a circle of radius 1 ft. 3 in. by a chord which subtends at the centre an angle of 23° 40′. [$\sin 23^\circ\, 40' = {\cdot}4014$.]

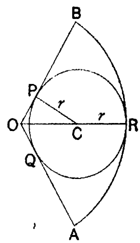
11.* A regular pentagon is inscribed in a circle of radius 10 inches: find the area of the minor segment cut off from the circle by one of the sides. [$\sin 72^\circ = {\cdot}9511$.]

12.* ABCD is a square on a side of 1 inch, and from A and C as centres, with AB, CB as radii, two arcs are described from B to D: find the area of the curvilinear figure included between the two arcs.

*SECTION III.

PROBLEMS ON CIRCLES, SECTORS, AND SEGMENTS.

58. The following examples are intended chiefly as a geometrical exercise. Some of the questions require the solution of an algebraical equation and a little work in elementary Trigonometry: all will be found to depend on principles proved in Euclid's Third Book.



Example i. Find the area of a circle inscribed in a sector whose angle is 120°, and whose radius is 10 inches.

In the adjoining figure C is the centre of the inscribed circle, which touches the radii of the sector at P and Q, and the arc at R.

Let r be the radius of the inscribed circle.

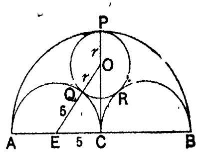
Then $\quad \mathsf{OR} = \mathsf{OC} + \mathsf{CR},$

that is, $\quad 10 = \mathsf{CP} \operatorname{cosec} 60° + r,$

or, $\quad 10 = r \operatorname{cosec} 60° + r.$

This equation determines r; hence the area of the circle is found.

Example ii. AB is a line 20 inches in length, and C is its middle point. On AB, AC and CB semicircles are described. Find the radius of the circle inscribed in the space enclosed by the three semicircles.



In the accompanying figure O is the centre of the inscribed circle which touches the three semicircles at P, Q and R.

Then OQ passes through E, the centre of the semicircle AC. [Euc. III. 12.]

Let r be the radius of the inscribed circle.

Then $$\mathsf{CP} = \mathsf{OP} + \mathsf{CO} = \mathsf{OP} + \sqrt{\mathsf{OE}^2 - \mathsf{EC}^2};$$

$$\therefore\ 10 = r + \sqrt{(r+5)^2 - 5^2}.$$

Hence $$(10 - r)^2 = (r+5)^2 - 5^2.$$

From which $$r = 3\tfrac{1}{3} \text{ inches.}$$

*EXAMPLES. VIII. C.

PROBLEMS ON CIRCLES, SECTORS, AND SEGMENTS.

[*Higher Course.*]

Note. $\pi = 3{\cdot}1416$; $\sqrt{2} = 1{\cdot}4142$; $\sqrt{3} = 1{\cdot}7320$.

1. A regular hexagon is inscribed in a circle of radius 10 inches: find (in square inches correct to two decimal places) the area of one of the segments lying between the circle and the hexagon.

2. From the angular points of a square as centres four circles are described. If the side of the square is 8 inches, and the radius of each circle is 4 inches, find the area of the curvilinear figure included between the circles.

3. Two equal circles each of radius 9 inches touch each other externally, and a common tangent (direct) is drawn to them. Find the area of the space included between the circles and the tangent.

4. Three circles of radius 1 foot are placed so that each touches the other two. Find (to the nearest square inch) the area of the curvilinear figure included between them.

5. From the angular points of a regular hexagon as centres six equal circles are described. If the side of the hexagon is 10 inches and the radius of each circle is 5 inches, find the area of the figure enclosed between the circles.

6. Two equal circles of radius 5 inches are described so that the centre of each is on the circumference of the other. Find the area of the curvilinear figure intercepted between the two circumferences.

7. Two equal circles of radius 5 inches intersect so that their common chord is equal to their radius; find (in square inches to two decimal places) the area of the curvilinear figure intercepted between the two circumferences.

8. Find the circumference of a circle inscribed in a quadrant of a circle whose radius is 8 inches. Give the result in inches, correct to two places of decimals.

9. Find the area of a circle inscribed in a sector whose angle is 60° and whose radius is 15 inches.

10. Two equal circles of radius 26 inches touch one another externally and a direct common tangent is drawn. Find the area of the circle inscribed between the two given circles and the tangent.

11. Two tangents at right angles to one another are drawn to a circle of radius 14 inches: find the area of the figure enclosed between the tangents and the circumference.

12. Two tangents making an angle 60° with one another are drawn to a circle of radius 3 inches: find the area of the figure enclosed between the tangents and the circumference.

13. An arc of 90° is cut off from a circle by a chord 16 inches in length. Find the area of the greatest circle that can be inscribed in the segment so formed.

14. An arc of 60° is cut off by a chord from a circle of radius 7 inches. Find the circumference of the greatest circle that can be inscribed in the segment so formed.

15. Two tangents perpendicular to one another are drawn to a circle of radius 10 inches. Find the radii of the two circles which touch the given circle and the tangents; and find the area of the smaller of these two circles.

16. If the two tangents in Question 15 were inclined at an angle of 120° to one another, find the radii of the two circles which may be drawn to touch the given circle and the tangents.

17. Two equal circles of diameter 9 inches touch one another, and from the point of contact as centre a third circle of radius 9 inches is drawn. Find the radius and the area of the circle inscribed in either of the two spaces enclosed by the three given circles.

18. Three equal circles of radius 10 inches are drawn so that each touches the other two. Find the radius (i) of the circle circumscribed about the three given circles, and (ii) of the circle inscribed in the space between them.

CHAPTER IX.

SIMILAR FIGURES

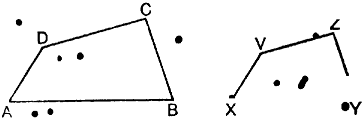
59. **Similar figures** may be described as figures of the same *shape*, but not necessarily of the same *size*.

Thus *circles* of all sizes are similar figures.

Sectors of circles whose radii include equal angles are similar.

Again the irregularly curved boundary of a county and its representation on a map are similar figures.

60. *Rectilineal* figures are similar, or of the same shape, when corresponding angles are equal, and corresponding lines proportional.

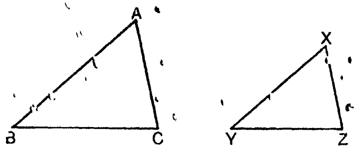


For instance the figures ABCD, XYZV are similar if

(i) the angles at A, B, C, D are equal respectively to the angles at X, Y, Z, V; and if

(ii) AB : XY = BC : YZ = CD : ZV = DA : VX.

61. Thus similarity in rectilineal figures includes two distinct properties. Two rectilineal figures of more than three sides might possess one of these properties and not the other, but they would not then be of the same shape. In the case of *triangles*, however, Euclid shews [VI. 4, 5] that if one element of similarity exists, the other must necessarily exist also.



That is to say, in the triangles ABC, XYZ if the angles at A, B, C are respectively equal to the angles at X, Y, Z, then

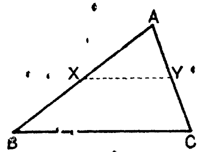
$$AB : XY = BC : YZ = CA : ZX. \quad \text{[Euc. VI. 4.]}$$

And if the triangles are such that

$$AB : XY = BC : YZ = CA : ZX,$$

then the angles at A, B, C are respectively equal to the angles at X, Y, Z. [Euc. VI. 5.]

62. The adjoining figure illustrates a case of frequent occurrence.



Here ABC is a triangle and XY is drawn parallel to the base BC.

Then it is clear that the angles of the triangle ABC are severally equal to those of the triangle AXY [Euc. I. 29];

hence $AB : AX = BC : XY = AC : AY$. [Euc. VI. 4.]

63. Another important proportion arises from the figure of the last article. Euclid proves [VI. 2] that a straight line drawn parallel to a side of a triangle divides the other two sides proportionally. That is to say,

$$AX : XB = AY : YC.$$

Also $$AB : BX = AC : CY.$$

64. We will now show how these principles may be applied.

Example i. In the figure of Art. 62, the sides of the triangle ABC are known. AB = 51 inches, BC = 20 inches, CA = 37 inches. If AX measures 34 inches, find the other sides of the triangle AXY.

We have $\quad XY : BC = AX : AB,$

$$\frac{XY}{20} = \frac{34}{51}.$$

$$\therefore \; XY = \frac{34 \times 20}{51} \text{ inches} = 13\tfrac{1}{3} \text{ inches}.$$

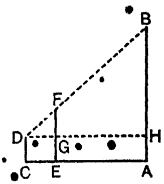
Again, $\quad AY : AC = AX : AB,$

or
$$\frac{AY}{37} = \frac{34}{51}.$$

$$\therefore \; AY = \frac{34 \times 37}{51} = 24\tfrac{2}{3} \text{ inches}.$$

Example ii. A man, wishing to ascertain the height of a tower, fixes a staff vertically in the ground at a distance of 27 feet from the tower. Then, retiring 3 feet farther from the tower he sees the top of the staff in line with the top of the tower. If the observer's eye and the top of the staff are respectively 5 ft. 4 in. and 12 feet above the ground, find the height of the tower.

In the adjoining figure AB represents the tower, EF the staff, and CD the observer in his final position.



DGH is a horizontal line drawn from the observer's eye.

Then $\quad DC = GE = HA = 5\tfrac{1}{3}$ feet;

$\quad HD = 30$ feet; $FG = 6\tfrac{2}{3}$ feet;

$\quad GD = 3$ feet.

And $\quad BH : HD = FG : GD,$

or
$$\frac{BH}{30} = \frac{6\frac{2}{3}}{3}.$$

$$\therefore \; BH = 66\tfrac{2}{3} \text{ feet}.$$

Hence the height of the tower = $66\tfrac{2}{3} + 5\tfrac{1}{3} = 72$ feet.

65. *The areas of similar rectilineal figures are proportional to the* squares *of corresponding sides.* Euc. VI. 20.

The areas of similar curvilinear figures are proportional to the squares *of any corresponding lines that may be drawn in them.*

For example, the areas of circles are proportional to the squares of their diameters.

Example i. The sides of a triangle are 21 inches, 20 inches, and 13 inches. Find the area of a similar triangle whose sides are to the corresponding sides of the first in the ratio 25 : 3.

$$Area\ of\ given\ triangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{27 \times 6 \times 7 \times 14}$$
$$= 9 \times 2 \times 7 \text{ sq. inches.}$$

And *required triangle* : *given triangle* $= 25^2 : 3^2$.

$$\therefore\ required\ triangle = \left(\frac{25}{3}\right)^2 \times given\ triangle$$
$$= \frac{25 \times 25}{9} \times 9 \times 2 \times 7 \text{ sq. inches}$$
$$= 8750 \text{ sq. inches.}$$

Example ii. In a survey map an estate of 144 acres is represented by a quadrilateral ABCD. If the diagonal AC is 6 inches, and the perpendiculars from B and D on AC are 1·8 inches and ·9 inch respectively, on what scale was the map drawn?

$$\text{Area of plan ABCD} = \frac{1}{2}\text{AC} \times (\text{sum of offsets})$$
$$= \frac{1}{2} \times 6 \times 2{\cdot}7 \text{ sq. in.} = 8{\cdot}1 \text{ sq. in.}$$

Hence 8·1 sq. inches represent 1440 sq. chains.

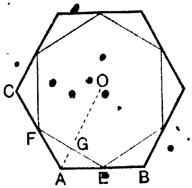
$$\therefore\ 1 \text{ sq. inch represents } \frac{1440}{8{\cdot}1} = \frac{1600}{9} \text{ sq. chains.}$$

$$\therefore\ 1\ \textit{linear}\ \text{inch represents } \frac{40}{3} \text{ chains} = \frac{1760}{6} \text{ yards.}$$

That is, the scale is six inches to the mile.

**Example* iii. Within a given regular hexagon, drawn on a side of 40 inches, a second hexagon is inscribed by joining the middle points of the sides taken in order. Find the area of the inscribed figure.

In the diagram, AB, AC are consecutive sides of a hexagon, and EF the side of the inscribed hexagon obtained by joining E, F the middle points of AB, AC.



Then $\frac{EF}{AB} = \frac{2 \cdot EG}{2 \cdot AE} = \frac{EG}{AE} = \sin 60°$

$$= \frac{\sqrt{3}}{2}.$$

And $\frac{\text{inscribed hexagon}}{\text{given hexagon}} = \left(\frac{EF}{AB}\right)^2 = \frac{3}{4}$

But the area of the given hexagon, being six times the area of an equilateral triangle on the same side, may be found by Art. 25 to be 259·81 sq. inches.

∴ the *area of the inscribed hexagon* $= \frac{3}{4} \times 259 \cdot 81$ sq. inches

$= 194 \cdot 85$ sq. inches.

EXAMPLES. IX.

On Similar Figures.

[Elementary and Higher Course.]

(Sides and Lines.)

1. The triangle ABC is right-angled at C; and from P, a point in the hypotenuse AB, PQ is drawn parallel to BC.

(i) If AP = 21 inches, PB = $3\frac{1}{2}$ inches, and AQ = 18 inches: find QC.

(ii) If AQ = 1·25 inches, AP = 2·25 inches, and QC = 10 inches: find PB.

(iii) If AC = 8·75 inches, PB = 2·25 inches, and QC = 1·75 inches: find AP.

(iv) If AB = 17·6 inches, AC = 14·4 inches, and QC = 5·4 inches: find AP.

(v) If BC = $12\frac{1}{2}$ inches, AC = 30 inches, and PQ = 5 inches: find AQ and AP.

(vi) If BC = 12 inches, QC = 28 inches, and PQ = 2·4 inches: find AQ and AB.

(vii) If BC = 7 inches, PB = 20 inches, and PQ = 1·4 inches: find AP and AC.

(viii) If BC = 12 inches, AQ = 7 inches, and QC = 28 inches: find PB.

(ix) If AP = 5 ft. 8 in., PB = 11 ft. 4 in., and PQ = 2 ft. 8 in.: find QC.

2. A man, wishing to ascertain the width of an impassable canal, takes two rods 3 feet and 5 feet in length. The shorter he fixes vertically on one bank, and then retires at right angles to the canal, until on resting the other rod vertically on the ground, he sees the ends of the two rods in a line with the remote bank. The distance between the rods he finds to be 60 feet. What is the width of the canal?

3. A man, wishing to ascertain the height of a tower, fixes a rod 11 feet in length vertically in the ground at a distance of 80 feet from the tower. On retiring 10 feet further from the tower, he sees the top of the rod in a line with the top of the tower. If the observer's eye is $5\frac{1}{2}$ feet above the ground, find the height of the tower.

4. A man whose height is 6 feet, standing 32 feet from a lamp-post, observes that his shadow cast by the light at the top is 8 feet in length. How high is the light above the ground, and how long would be the shadow of a boy 5 feet in height standing 20 feet from the post?

5. A man 6 feet in height, standing 15 feet from a lamp-post, observes that his shadow cast by the light at the top is 5 feet in length: how long would his shadow be if he were to approach 8 feet nearer to the post?

6. The triangle ABC is right-angled at C; and CA′ is drawn perpendicular to AB, A′C′ is drawn from A′ perpendicular to BC. If AC = 5 inches, and BC = $6\frac{2}{3}$ inches; find CA′ and A′C′.

7. In the triangle ABC, AB = 9 inches, BC = 8 inches, and CA = 7 inches. In AB a point P is taken 2 inches from A, and PQ is drawn parallel to BC. Find the lengths of PQ and QC.

8. In a triangle ABC, AD is the perpendicular from A on BC; and through X, a point in AD, a line is drawn parallel to BC meeting the other sides in P, Q. If BC = 9 inches, AD = 8 inches and DX = 3 inches. Find the length of PQ.

9. In the triangle ABC, AD is the perpendicular drawn from A to BC; and through X, a point in AD, a line is drawn parallel to BC, meeting the other sides in P, Q. If AB = 13 inches, BC = 14 inches, CA = 15 inches, and DX = 3 inches, find the sides of the triangle APQ.

10. With the figure of Question 9, if AB = 29 inches, BC = 36 inches, CA = 25 inches, and DX = 4 inches, find the sides of the triangle APQ.

11.* A person standing due South of a light-house observes that his shadow, cast by the light at the top, is 24 feet long. On walking 300 feet due East, he finds his shadow to be 30 feet. Supposing him to be 6 feet high, find the height of the light above the ground.

12.* A person standing due South of a light-house observes that his shadow, cast by the light at the top, is 23 feet long. On walking 240 feet due East he finds his shadow to be 28 ft. 9 in. Supposing his height to be 5 ft. 9 in., find the height of the light above the ground.

(*Areas.*)

13. The area of an equilateral triangle on a base of 10 inches is approximately 43·3 square inches; find the area of an equilateral triangle on a base of 2 inches.

14. The area of a regular hexagon on a side of 12 inches is approximately 374·12 square inches; find the area of a similar figure on a base of 2 inches.

15. The sides of a triangle are 25 feet, 17 feet and 12 feet. Find the area of another triangle whose sides are respectively one-third of those of the first.

16. ABC and XYZ are two similar triangles whose areas are respectively 245 and 5 square inches. If AB = 21 inches, find XY.

17. ABC is a triangle right-angled at C; and from X, a point in AB, XY is drawn parallel to BC. If $AB = 17$ inches, $BC = 15$ inches, and $AY = 2$ inches, find the area of the triangle AXY.

18. ABC is a triangle right-angled at C; and from X, a point in AB, XY is drawn parallel to BC. If $AC = 7$ inches, $BC = 24$ inches, and $AX = 5$ inches, find the area of the triangle AXY.

19. In the triangle ABC, $AB = 29$ inches, $BC = 36$ inches, $CA = 25$ inches. From a point X in AB a line XY is drawn parallel to BC. If $AX = 5{\cdot}8$ inches, find the area of the triangle AXY.

20. In the triangle ABC, $AB = 21$ chains, $BC = 17$ chains, $CA = 10$ chains. From a point X in AB produced, a line XY is drawn parallel to BC to meet AC produced at Y. If $AX = 105$ chains, find the acreage of the triangle AXY.

(*Plans and Scales.*)

21. The ground-plan of a house drawn to a scale of one inch to 15 feet is represented by a rectangle 8 inches by 6 inches. What area is covered by the basement of the house?

22. In the plan of an estate drawn to the scale of six inches to the mile, a field is represented by a square on a side of ·75 inch. Find the acreage of the field.

23. What is the perimeter of a field if in the ordnance survey map of 25 inches to the mile it is represented by a square containing 6·25 square inches?

24. In a plan, drawn to the scale of 22 inches to the mile, a plot of ground is represented by a quadrilateral ABCD. If AC is 11 inches, and the perpendiculars drawn from B and D on AC are respectively 6 inches and 5 inches, find the rent of the ground at £2. 10*s.* an acre.

25. A field of 9 acres is represented in a plan by a triangle whose sides are 25, 17 and 12 inches. On what scale is the plan drawn, and what length will be represented by 80 inches?

26. In a survey map an estate of 512 acres is represented by a quadrilateral ABCD. If AC is 20 inches, and the perpendiculars from B and D on AC are 24 and 26 inches respectively, on what scale was the map drawn? Give the result in inches to the mile.

[*Higher Course.*]

(*Geometrical Problems.*)

27. A triangle ABC is divided into two equal parts by a straight line XY drawn parallel to the base BC. If AB = 100 inches, find AX.

28. A triangle ABC is divided into three equal parts by two lines XX′ and YY′ drawn parallel to BC. If AB = 100 inches, find AX and AY.

29. In the triangle ABC, AB = 13 inches, BC = 14 inches, AC = 15 inches. PQ is drawn parallel to BC, cutting the other sides at P and Q. If the perpendicular distance between PQ and BC is 3 inches, find the area of the triangle APQ.

30. In the triangle ABC, AB = 25 feet, BC = 29 feet, AC = 36 feet, XY is drawn parallel to AC cutting the other sides at X and Y. If the perpendicular distance between XY and AC is 4 feet, find the area of the triangle BXY.

31. In a given square, a square is inscribed by joining in order the middle points of the sides. In the inscribed square another square is inscribed in a similar way, and so on. If the side of the given square is 16 inches, find the area of the first inscribed square, also of the eighth inscribed square.

32. In a given triangle a similar triangle is inscribed by joining the middle points of the sides. In this inscribed triangle another similar triangle is inscribed in like manner, and so on. What fraction of the given triangle is the area of the sixth inscribed triangle?

33. In a circle of radius 32 inches an equilateral triangle is inscribed, and in this triangle a circle. In this circle an equilateral triangle is again inscribed, and in the triangle a circle. If this process is continued, find the area of the fourth circle (counting from the given one), and find which of the circles has an area of $3\frac{1}{7}$ square inches.

34. In a circle of radius 16 inches, a regular hexagon is inscribed, and in this hexagon a circle. In this circle a regular hexagon is again inscribed, and in the hexagon a circle. If this process is continued, find the area of the fifth circle (counting from the given one), and of the fourth hexagon.

CHAPTER X.

ON REGULAR POLYGONS.

66. DEFINITIONS:

(i) A **polygon** is a rectilineal figure of more than four sides.

A polygon of	*five*	sides is called a		**Pentagon,**
,,	*six*	sides	,,	**Hexagon,**
,,	*seven*	sides	,,	**Heptagon,**
,,	*eight*	sides	,,	**Octagon,**
,,	*nine*	sides	,,	**Nonagon,**
,,	*ten*	sides	,,	**Decagon,**
,,	*eleven*	sides	,,	**Undecagon,**
,,	*twelve*	sides	,,	**Dodecagon,**
,,	*fifteen*	sides	,,	**Quindecagon.**

(ii) A polygon is said to be **regular** when its sides are all equal, and its angles are all equal.

(iii) A circle is said to be **circumscribed** about a polygon when the circumference of the circle passes through the angular points of the polygon.

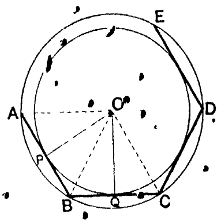
(iv) A circle is said to be **inscribed** in a polygon when the circumference touches each side of the polygon.

67. By the method of Euc. IV. 13 it may be shewn that

(i) *the bisectors of the angles of a regular polygon meet at a point:*

(ii) *the perpendiculars drawn from this point to the sides are all equal; and the lines joining this point to the vertices are all equal:*

(iii) *hence the point at which the bisectors of the angles intersect is the centre both of the inscribed and circumscribed circles.*



For example, let AB, BC, CD, DE be consecutive sides of a regular polygon of n sides; and let a be the length of its side, and r, R the radii of the inscribed and circumscribed circles: then

(i) the bisectors of the angles at A, B, C, D, ... meet at a point O.

(ii) OA = OB = OC = ... = R.

(iii) OP (the perp^r. from O on AB) = OQ = ... = r.

(iv) The triangles AOB, BOC, .. are all equal.

(v) Each of the angles AOB, BOC ... = the n^{th} part of four right angles, or $\dfrac{360°}{n}$.

68. *To find the area of a regular polygon of* n *sides, having given the length of the side and of the perpendicular from the central point on a side.*

$$\text{The area of the polygon} = n \times \text{area of } \triangle OBC$$
$$= n \times \tfrac{1}{2} BC \,.\, OQ$$
$$= \frac{n}{2} \times \textit{side} \times \textit{perpendicular}.$$

69. *To find the area of a regular polygon of from 5 to 12 sides, having given*

(i) *the length* (a) *of one side;*

or (ii) *the radius* (r) *of the inscribed circle;*

or (iii) *the radius* (R) *of the circumscribed circle.*

In the Table of the next article refer to the numbers corresponding to the assigned type of polygon.

The area of the polygon is found by multiplying

a^2 by the number given in column (i),

r^2 .... (ii),

R^2 .... (iii).

Example. Find the area of a regular octagon on a side of 10 inches.

The number corresponding to an *octagon* in column (i) of the Table is 4·82843.

$$\therefore \text{ area of octagon} = 10^2 \times 4{\cdot}82843$$
$$= 482{\cdot}843 \text{ sq. inches.}$$

*70. *To find formulæ for the area of a regular polygon of* n *sides in terms of* (i) *a side,* (ii) *the radius of the inscribed circle,* (iii) *the radius of the circumscribed circle.*

With the figure and notation of Art. 66, we have

$$OQ = BQ \cot BOQ,$$

or
$$r = \frac{a}{2} \cot \frac{180^\circ}{n}.$$

(i) The area of the polygon $= n \cdot \frac{1}{2} BC \cdot OQ$

$$= \frac{n}{2} \cdot a \cdot \frac{a}{2} \cot \frac{180^\circ}{n}$$

$$= a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}.$$

(ii) The area of the polygon $= n \cdot \frac{1}{2} OQ \cdot BC$

$$= \frac{n}{2} \cdot r \cdot 2r \tan \frac{180^\circ}{n}$$

$$= r^2 \times n \tan \frac{180^\circ}{n}.$$

(iii) The area of the polygon $= n \cdot \triangle BOC$

$$= n \cdot \tfrac{1}{2} OB \cdot OC \sin BOC$$

$$= \frac{n}{2} \cdot R^2 \cdot \sin \frac{360^\circ}{n}$$

$$= R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n}.$$

The values of the multipliers

$$\frac{n}{4}\cot\frac{180^\circ}{n},\quad n\tan\frac{180^\circ}{n},\quad \frac{n}{2}\sin\frac{360^\circ}{n}$$

are tabulated below for regular polygons of from 5 to 12 sides inclusive.

Table for calculating the area of a regular polygon, having given (i) a side, (ii) the radius of the inscribed circle, (iii) the radius of the circumscribed circle.

Name of Polygon	No. of sides n	(i) Multiply a^2 by $\frac{n}{4}\cot\frac{180^\circ}{n}$	(ii) Multiply r^2 by $n\tan\frac{180^\circ}{n}$	(iii) Multiply R^2 by $\frac{n}{2}\sin\frac{360^\circ}{n}$
Pentagon	5	1·72048	3·63271	2·37764
Hexagon	6	2·59808	3·46410	2·59808
Heptagon	7	3·63391	3·37100	2·73641
Octagon	8	4·82843	3·31371	2·82843
Nonagon	9	6·18182	3·27573	2·89254
Decagon	10	7·69421	3·24920	2·93893
Undecagon	11	9·36564	3·22994	2·97352
Dodecagon	12	11·19615	3·21539	3·00000

71. The following example will illustrate the method of Art. 68.

Example. Find the area of the circle inscribed in a regular decagon whose side is 14 inches.

Referring to the figure of Art. 67,

$$OQ = BQ \cot BOQ.$$

or,
$$r = 7\cot 18^\circ.$$

$$\therefore \text{ Area of inscribed circle} = \pi r^2 = \frac{22}{7}\times 7^2 \cot^2 18^\circ$$

$$= 22\times 7\times\frac{5+\sqrt{5}}{3-\sqrt{5}}$$

$$= 22\times 7\times(5+2\sqrt{5}) = 1458\cdot 71 \text{ sq. inches nearly.}$$

EXAMPLES. X.

On Regular Polygons.

[*Elementary Course.*]

1. Find the area of a regular pentagon on a side of 5 inches, giving the result in square inches true to the second decimal place.

2. Find the area of a regular decagon on a side of 8 inches. Give the result to the nearest hundredth of a square inch.

3. Find the area of a regular heptagon inscribed in a circle of radius 6 inches.

4. Find the value (to the nearest penny) of a regular hexagonal sheet of metal on a side of 4 feet, at the rate of 12*s.* 3*d.* a square yard.

5. If it costs £16. 13*s.* 4*d.* to fence in a regular octagonal enclosure at the rate of 12*s.* 6*d.* a yard, what would it cost to pave it at 3*s.* 4*d.* a square foot?

6. Find to the nearest hundredth of an inch the radius of a circle whose area is equal to that of a regular hexagon on a side of 11 inches. ($\pi = \frac{22}{7}$.)

[*Higher Course.*]

7. Find to the nearest hundredth of an inch the perpendicular distance between the opposite sides of a regular hexagon whose side is 10 inches.

8. A regular octagon is formed by cutting off the corners of a square whose side is 8 inches. Find the side of the octagon.

9. Find to the nearest hundredth of an inch the side of a dodecagon inscribed in a circle of radius 12 inches.

10. Find the area of a circle circumscribed about a regular octagon on a side of 7 inches. ($\pi = \frac{22}{7}$.)

11. The area of a dodecagon is 300 square inches; find the radius of the circle circumscribed about it.

12. Find the area of the circular ring bounded by the inscribed and circumscribed circles of a regular hexagon, the length of whose side is 20 inches. ($\pi = 3{\cdot}1416$.)

And shew that the area of the circular ring bounded by the inscribed and circumscribed circles of a regular polygon on a given side is the same whatever be the number of sides.

13. About a hexagonal enclosure whose side is 14 feet runs a path of uniform width. If the width of the path is 3 feet, find its area.

14. The basin of a fountain is a regular polygon of n sides, each side being a feet in length; and round it runs a path of uniform width. Find a trigonometrical expression to give the area of the path, if its width is b feet.

15. In a circle of radius 7 inches a regular octagon is inscribed, and a circle is inscribed in the octagon. Find the area of the inscribed circle to the nearest hundredth of a square inch. ($\pi = \frac{22}{7}$.)

16. The difference between a regular dodecagon and a hexagon on the same base is 113·4 square inches. Find the length of the base. [Take $\sqrt{3} = 1{\cdot}732$.]

17. The difference between the areas of a regular octagon and a square inscribed in the same circle is 82·8 square inches. Find the radius of the circle. [Take $\sqrt{2} = 1{\cdot}414$.]

18. Calculate the area of the square formed by joining the middle points of the alternate sides of a regular octagon on a side of 8 inches.

19. Calculate the area of the equilateral triangle formed by joining the middle points of the alternate sides of a regular hexagon whose side is 40 inches.

20. A regular polygon of n sides is formed by joining the middle points of alternate sides of a polygon of $2n$ sides. If the side of the latter polygon is a inches, find an expression to give (i) the side of the former polygon, (ii) its area, (iii) the area of its inscribed circle.

21. In a circle of radius 10 inches, a regular hexagon is inscribed, and a circle is inscribed in the hexagon. Within this circle a second hexagon is drawn, and in the hexagon a circle. If this process is continued, ad infinitum, find the sum of the areas of all the hexagons so drawn.

22. In a circle of radius r, a regular polygon of n sides is inscribed, and a circle is inscribed in the polygon. In this circle a second regular polygon is drawn, and in the polygon a circle. If this process is continued, ad infinitum, find the sum of the areas of all the circles.

CHAPTER XI.

IRREGULAR RECTILINEAL FIGURES. THE FIELD-BOOK.

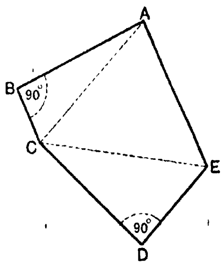
SECTION 1.

AREAS.

72. RECTILINEAL figures may always be divided into triangles and quadrilaterals whose area can be separately found by the methods of Chapters IV. and V. The sum of the results so obtained will be the area of the given figure.

Example i. Calculate the area of the figure ABCDE from the following data. The angles at B and D are right angles:

AB = 12 inches, DE = 9 inches,
BC = 5 inches, EA = 14 inches,
CD = 12 inches.



Here $AC = \sqrt{12^2 + 5^2} = 13$ inches;

and $CE = \sqrt{12^2 + 9^2} = 15$ inches.

(i) The $\triangle ABC = \frac{1}{2} . AB \times BC$

$= \frac{1}{2} . 12 \times 5 = 30$ sq. in.

(ii) The $\triangle CDE = \frac{1}{2} . CD \times DE = \frac{1}{2} . 12 \times 9 = 54$ sq. in.

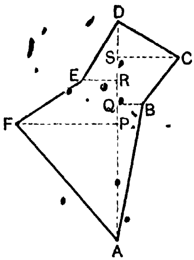
(iii) The $\triangle ACE = \sqrt{s(s-a)(s-b)(s-c)} = 84$ sq. in.

$\therefore$ the area of the figure ABCDE = 168 sq. inches.

73. In practice irregular rectilineal figures are most commonly sub-divided by means of a **base-line** (or **chain-line**) and **offsets** from it. The diagram given below will illustrate the method.

Two vertices (A and D in the figure) are chosen as **stations**, and joined by a line called the *base-line*. From the remaining vertices of the figure perpendiculars (known as **offsets**) are drawn to the base-line. The figure is thus divided into *right-angled triangles* and *right-angled trapeziums*, whose areas may be separately found if we know the lengths of the offsets and of the various sections of the base-line.

Example. Find the area of the enclosure of which the adjoining diagram is a plan. The following measurements have been made:



AD = 1675 links,	
AP = 900 links,	PF = 850 links,
AQ = 1040 links,	QB = 200 links,
AR = 1200 links,	RE = 250 links,
AS = 1880 links,	SC = 500 links.

Here we have to find the areas of the four right-angled triangles APF, AQB, DRE, DSC, and of the two right-angled trapeziums PREF, QSCB. It will be simpler to find *twice* the area of each of these figures, and to take half of the final result:

(i) Twice triangle APF = AP × PF = 900 × 850 sq. links
= 765000 sq. links.

(ii) Twice triangle AQB = AQ × QB = 1040 × 200 sq. links
= 208000 sq. links.

DR = AD − AR = 475 links.

(iii) Twice triangle DRE = DR × RE = 475 × 250 sq. links
= 118750 sq. links.

DS = AD − AS = 295 links

(iv) Twice triangle DSC = DS × SC = 295 × 500 sq. links
= 147500 sq. links.

PR = AR − AP = 300 links.

(v) Twice trapezium PREF = PR (PF + RE) = 300 × 1100 sq. links
= 330000 sq. links.

QS = AS − AQ = 340 links.

(vi) Twice trapezium QSCB = QS (QB + SC) = 340 × 700 sq. links
= 238000 sq. links.

By addition, twice the whole figure = 1807250 sq. links;

∴ the required area = 903625 sq. links = 9·03625 acres
= 9 ac. 0 r. 6 p. nearly.

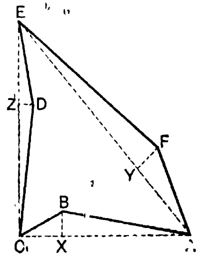
74. When the given figure approximates in shape to a triangle or quadrilateral, more than one chain-line is introduced; and **insets,** as well as offsets, are used, in the manner illustrated by the figure given below.

Here the irregular figure ABCDEF approximates to the form of a right-angled triangle EAC, the area of which must first be calculated. From this area the triangles derived from the *insets* XB, ZD must be subtracted; and the triangle derived from the offset YF must be added to the result.

Example. Calculate the area of the enclosure ABCDEF from the accompanying plan, having given the following measurements:

the angle ACE is a right angle;

AC = 650 links,
CE = 720 links,
BX = 90 links,
DZ = 50 links,
FY = 120 links.



Here $EA = \sqrt{CA^2 + CE^2}$

$= \sqrt{(650)^2 + (720)^2}$

$= 970$ links.

(i) Twice triangle ACE = AC × CE = 650 × 720 sq. links
= 468000 sq. links.

(ii) Twice triangle ABC = AC × BX = 650 × 90 sq. links
= 58500 sq. links.

(iii) Twice triangle CDE = CE × DZ = 720 × 50 sq. links
= 36000 sq. links.

(iv) Twice triangle EFA = EA × FY = 970 × 120 sq. links
= 116400 sq. links.

Hence *adding* the results of (i) and (iv), and *subtracting* the results of (ii) and (iii) we have

twice required figure = 489900 sq. links = 4·899 acres;

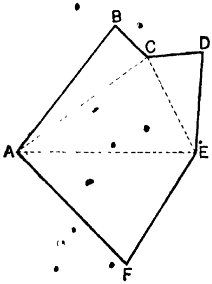
∴ required area = 2·4495 acres

= 2 ac. 1 r. 32 p. nearly.

EXAMPLES. XI. A.

IRREGULAR RECTILINEAL FIGURES.

[Elementary and Higher Course.]



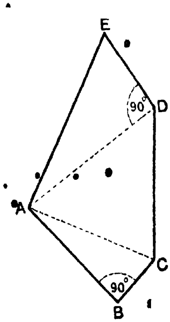
1. In the irregular hexagon ABCDEF, it is given that the angles at B, C and F are right angles, and

AB = 35 feet, DE = 16 feet,

BC = 12 feet, EF = 24 feet,

CD = 12 feet, FA = 45 feet.

Find the area in square feet.



2. Calculate the area of the figure ABCDE from the following data.

AB = 12 in.; the ∠ ABC a right angle;

BC = 5 in.,

CD = 14 in.,

AD = 15 in.; the ∠ ADE a right angle;

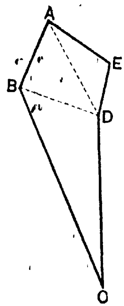
DE = 8 in.

Give the result in square feet and square inches.

3. Calculate the area of the figure ABCDE from the following measurements :

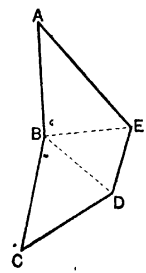
AB = 13 ft.,	EA = 1[illegible] ft.,
BC = 51 ft.,	AD = 21 ft.,
CD = 37 ft.,	BD = 20 ft.,
DE = 10 ft.	

Give the result in square yards and feet.



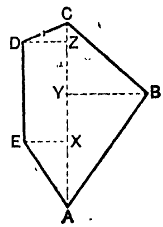
4. Calculate the acreage of the figure ABCDE from the following measurements :

AB = 290 links,	EA = 360 links,
BC = 300 links,	BE = 250 links,
CD = 280 links,	BD = 260 links,
DE = 170 links.	

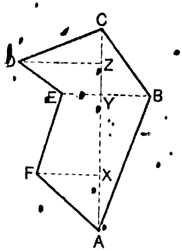


5. Calculate the area of the figure ABCDE, from the following measurements, where XE, YB, ZD are off-sets from AC.

AX = 6 yds.,	XE = 4 yds.,
XZ = 8 yds.,	YB = 8 yds.,
ZC = 2 yds.,	ZD = 4 yds.



6. Calculate the area of the figure ABCDEF from the measurements given below; XF, YB, YE and ZD being off-sets from AC.



AC = 180 yds., XF = 60 yds.,

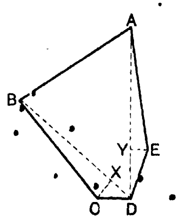
AX = 50 yds., YE = 40 yds.,

XY = 70 yds., YB = 50 yds.,

YZ = 30 yds., ZD = 80 yds.,

ZC = 30 yds.

7. Find in square yards the area of the figure ABCDE, having given



AB = 13 yards,

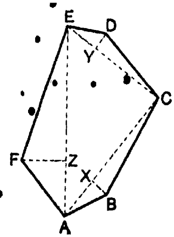
BD = 14 yards,

DA = 15 yards,

CX = 2 yards,

EY = 2 yards.

8. Calculate the area of a field, of which the adjoining diagram is a plan, from the following measurements.



AC = 2900 links, BX = 400 links,

CE = 2500 links, DY = 400 links,

EA = 3600 links, FZ = 950 links.

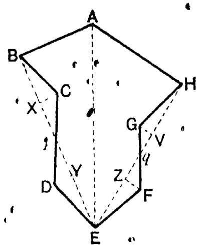
Give the result in acres, roods, and the nearest pole.

9. Calculate the area of the figure ABCDEFGH from the following data: the angles ABE, AHE are right angles;

AB = 7 yards,
BE = 24 yards,
EH = 20 yards,
HA = 15 yards;

p and q are the middle points of BE and EH;

CX = DY, and FZ = GV.



SECTION II.

THE FIELD-BOOK.

75. ONE of the most important practical uses to which the mensuration of plane figures is put is the survey of land and the calculation of areas from ascertained measurements. From the examples of the preceding section the student will have learned how a simple plot of ground with rectilineal boundaries may be sub-divided for this purpose into triangles and quadrilaterals; and what measurements are likely to be needed. Without entering upon the manner in which these measurements are actually made in practice (as knowledge of this kind is most readily gained in the field), we will explain how the surveyor records them in his **Field-Book**; and the beginner will find it a useful introduction to his practical work to draw plans from

field-book notes in a few simple cases, and to calculate the area of the ground represented.

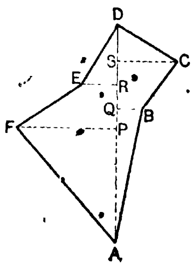
76. We will first consider a plot of ground surveyed by a single chain-line and offsets from it.

Example 1. Draw a plan and calculate the area of a field from the following notes:

The Field-Book.

	Links	
	to *D*	
	1675	
	1380	500 to *C*
to *E* 250	1200	
	1040	200 to *B*
to *F* 850	900	
From	*A*	go North

The Plan.



The field-book is to be read *upwards*, beginning at the lowest line, which tells us in this case that the selected chain-line runs from a station *A* due North.

The centre column of the field-book refers to measurements made *from A* along the chain-line to the points from which the offsets spring: the columns on the right and left refer to the lengths of offsets to the right and left of the chain-line.

Thus to draw a plan from the field-book, we first lay down the chain-line *AD* to represent a direction due North from *A*. In *AD* we take *AP* of sufficient length to represent 900 links on some fixed scale. From *P* we draw an offset *PF to the left* of such length as will represent 850 links on the same scale. The offsets *QB*, *RE*, *SC* are to be drawn in a similar manner: and then the point *D* is determined so that *AD* may represent 1675 links. The boundary *AFEDCBA* is now filled in, and the plan is complete.

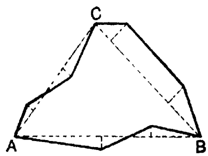
For the calculation of the area, see Art. 73, Ex. 1.

77. When a field is surveyed by means of a triangle, as in the example given below, three chain-lines are used, and the offsets from each are recorded in the manner already explained.

Example. Draw a plan of a field from the following notes:

The Field-Book.

The Plan.



	Links	
	to *A*	
	2500	
	1800	250
0	1450	
300	1100	
From	*C*	range to *A*
	to *C*	
	2900	
	2450	450
	[illegible]00	450
From	*B*	turn left to *C*
	to *B*	
	3600	
280	2650	
	2220	0
	1720	320
From	*A*	go East

The horizontal lines in the field-book separate the notes referring to the several chain-lines. Thus the field-book shews *three* chain-lines *AB*, *BC*, *CA*, which must first be drawn: their lengths are to represent respectively 3600 links, 2900 links and 2500 links. This triangle is drawn by the method of Euc. Bk. I. Prop. 22; and the entry "from *B* turn left to *C*" shews that the triangle is to be on the *left* of *AB*, looking from *A* to *B*.

The offsets from the chain-lines are now to be laid down as explained in the last Example, it being observed that the entry 0 (as an offset) indicates a point at which the boundary crosses the chain-line.

The area of the figure may now be calculated in the manner explained in Art. 74.

EXAMPLES XI. B

THE FIELD-BOOK.

[Elementary and Higher Course.]

1. Draw the plan and calculate the area of a field from the following notes:

	Links	
	to B	
	520	
to E 80	440	
	220	110 to C
to D 120	150	
From	A	go North

2. Draw the plan, and calculate the area of a field from the following notes:

	Links	
	to B	
	550	
	500	80 to D
	310	160 to C
to E 200	180	
From	A	go N.E.

3. Draw the plan, and calculate the area of a field from the following notes:

	Links	
	to B	
	1200	
	1000	200
80	825	
0	600	
40	350	220
From	A	go N. 30° W. to B

4. Draw the plan, and calculate the area of a field from the following field-book.

	Links	
	to D	
	720	
	650	400 to C
to E 30	625	
0	420	
	250	20 to F 425 to B
	0	0
From	A	go N.W.

5. Draw the plan, and calculate the area of a field from the following notes:

	Links	
	to (2)	
45	1440	1100
0	810	
45	560	1225
0	240	
45	0	1100
From	(1)	range North to (2)

6. Draw a plan and calculate the area of a field, surveyed by a right-angled triangle ABC, one side of which, CA, is a boundary of the field. The field-book is as follows:

	Links	
	to A	
	1700	
From	C	range to A
	to C	
	800	
	400	65
From	B	go North
	to B	
	1500	
	1100	180
	625	240
From	A	go East

Note. The line AC need not theoretically be measured, as the angle ABC is a right angle; but it is usual to measure more lines than are absolutely necessary to serve as a check on the accuracy of previous measurements and calculations.

7. Draw a plan, and calculate the area of a field, surveyed by means of a triangle ABC, from the following notes:

	Links	
	to A	
	370	
	300	40
	120	35
From	C	range to A
	to C	
	200	
	128	45
From	B	turn left
	to B	
	510	
	361	22
	250	0
	157	24
From	A	go N.E.

PART II.

THE MENSURATION OF SOLID FIGURES.

CHAPTER XII

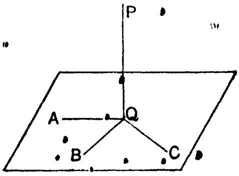
INTRODUCTORY.

78. Definitions. 1. **Parallel planes** are such as do not meet though produced.

The floor and ceiling of a room are parallel planes; so are each pair of opposite sides.

2. A straight line is said to be **perpendicular to a plane,** when it makes right angles with every straight line which meets it in that plane.

The meaning of this definition becomes clear, if we imagine a rod PQ fixed in the ground, and a number of straight lines such as QA, QB, QC, &c., drawn from the foot of the rod in various directions on the ground. Thus for PQ to be perpendicular to the ground, the angles PQA, PQB, PQC, &c., must be all right angles.

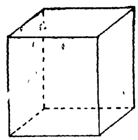


3. A **solid figure**, or **solid**, is that which has length, breadth, and thickness.

4. A solid is bounded by one or more surfaces.

If the surfaces bounding a solid are plane, they are called **faces**, and the bounding lines of the faces are called **edges**.

5. A **cube** is a solid bounded by six equal square faces



It will be seen from the figure that opposite faces of a cube are parallel, and that a cube has *twelve* equal edges.

6. A cube, of which each edge is one inch in length, is called a **cubic inch.** Similarly, if each edge of a cube measures one foot, or one yard, it is called a **cubic foot,** or **cubic yard.**

7. The **volume**, or content, of a solid figure is the space contained within its bounding surfaces.

The volume of a solid is measured by the number of times it contains some specified cubic unit, such as a cubic inch, or a cubic foot.

TABLES.

79. The following Tables are those chiefly used in the Mensuration of Solid Figures.

I. 1728 (or 12^3) cubic inches = 1 cubic foot,
27 (or 3^3) cubic feet = 1 cubic yard.

II. A cubic foot of pure water weighs nearly 1000 ounces Avoir., or more nearly $62\frac{1}{2}$ lbs.

The actual weight of a cubic foot of water (correct to the nearest thousandth of an ounce) is 997·137 ounces.

III. A gallon holds 10 lbs. Avoir. of pure water. Hence it may be calculated that

(i) 1 gallon contains nearly $277\frac{1}{4}$ cubic inches,

(ii) 1 cubic foot contains nearly $6\frac{1}{4}$ gallons.

The actual content of 1 gallon (correct to the nearest thousandth of a cubic inch) is 277·274 cubic inches. [See Prefatory Note.]

The Metric System.

In the Metric System the unit of length is the **metre**, approximately 39·3708 inches.

A *decimetre*, *centimetre*, *millimetre* are respectively $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ of a metre.

The unit of *volume* is the *cubic metre*.

The unit of *capacity* is the **litre**, or *cubic decimetre*. The unit of *weight* is the **gramme**, namely the weight of a cubic centimetre of pure water.

Note. Since 1 metre = 39·3708 inches, it may be found that

(i) 1 cubic metre = 35·31 cubic feet = 220·1 gallons

(ii) 1 litre = 61·027 cubic inches = ·2201 gallons.

And remembering that the weight of one gallon of water is 10 lbs. Avoir., or 70,000 grains; it may be calculated that the weight of one cubic centimetre of water, i.e. the *gramme*, is 15·432 grains.

Cube Root.

80. As certain questions in Solid Mensuration require the extraction of a cube root, an example of the method, fully worked out, is given below for reference.

Find the cube root of 381078·125

		381 078 ·125	72·5
		343	
$(70)^2 \times 3 =$	14700	38078	
$(70) \times 2 \times 3 =$	420		
$2^2 =$	4		
	15124	30248	
$(720)^2 \times 3 =$	1555200	7830125	
$720 \times 5 \times 3 =$	10800		
$5^2 =$	25		
	1566025	7830125	

Note. The periods, each consisting of three digits, are to be marked off from the *decimal point* each way. Thus to find the cube root of 19·02673, ·0526, ·000000045, we begin by marking off the periods as follows:—19·026 730, ·052 600, ·000 000 045. Observe that each period furnishes one digit to the cube root.

If more than three significant digits are required in the result, it is generally better to work the example by the Table of Logarithms, as illustrated in Chapter XXIII.

EXAMPLES. XII.

Find the cube roots of

1. 16·974593.
2. ·000012167.
3. ·000000125.
4. $3\frac{3}{4}$.
5. $18\frac{2}{11}$.
6. $150\sqrt{2}$.

CHAPTER XIII.

THE RECTANGULAR SOLID AND CUBE.

81. Definition. A **rectangular solid** is a body bounded by six rectangular faces, opposite faces being equal and parallel.

Fig. i.

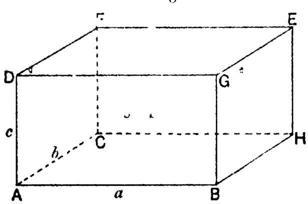


Fig. ii.

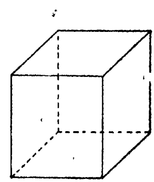


Fig. i. represents a rectangular solid, of which AB is the length, AC the breadth, and AD the height. It is bounded by six rectangular faces, of which the opposite faces ABGD, CHEF are equal and parallel: so also are the opposite faces ABHC and DGEF; and similarly the face ACFD is equal and parallel to the opposite face BHEG.

A rectangular solid which has its length, breadth, and height equal (as in fig. ii) is a cube.

Note. The name c\uold has lately been applied to a rectangular solid whose length, breadth and height are not all equal.

[Hayward's *Solid Geometry.*]

82. *To find the whole surface of a rectangular solid, we must add together the areas of the six rectangular faces.*

Thus if the length AB $= a$ units, the breadth AC $= b$, and the height AD $= c$,

then each of the faces ABHC, DGEF contains ab square units;

Similarly ABGD, CHEF each contain ac square units; and ACFD, BHEG each contain bc square units.

Hence (i) *the whole surface of the rectangular solid*

$$= 2ab + 2ac + 2bc \text{ sq. units}$$
$$= 2(ab + ac + bc) \text{ sq. units.}$$

If each edge of the cube $= a$ units, then each of its six faces contains a^2 square units;

hence (ii) *the whole surface of a cube* $= 6a^2$ square units.

If a room is a feet long, b feet wide, and c feet high, it follows that

the area of the four walls $= 2ac + 2bc$ square feet
$= 2c(a + b)$ square feet.

Example i. Find in square feet the whole surface of a rectangular block of stone whose length is 2 yds. 2 ft., bredth 1 yd. 1 ft., and height 9 inches.

Here $a = 8$ feet, $b = 4$ feet, and $c = \frac{3}{4}$ foot.

Surface of rectangular block $= 2(ab + ac + bc)$

$= 2(8 \times 4 + 8 \times \frac{3}{4} + 4 \times \frac{3}{4})$ sq. feet

$= 2(32 + 6 + 3)$ sq. ft. $= 82$ sq. feet.

Example ii. How many yards of paper 22 inches wide are required for the walls of a room 15 ft. 4 in. long, 14 ft. 8 in. wide, and 11 feet high?

Here $a = 15\frac{1}{3}$ ft., $b = 14\frac{2}{3}$ ft., $c = 11$ ft.

Area of four walls $= 2c(a + b)$

$= 2 \times 11 (15\frac{1}{3} + 14\frac{2}{3})$ sq. feet

$= 22 \times 30$ sq. ft. $= \dfrac{22 \times 10}{3}$ sq. yards.

But width of paper $= 22$ inches $= \frac{11}{18}$ yard.

$\therefore$ *length of paper required* $=$ *area of paper* $\div$ *width*

$$= \frac{22 \times 10}{3} \div \frac{11}{18} \text{ yds.} = 120 \text{ yards.}$$

Example iii. The whole surface of a cube is 5 sq. ft. 6 sq. in.; find he length of each edge.

The surface of the cube $= 6a^2$,

$\therefore \quad 6a^2 = 726$ sq. inches,

$\therefore \quad a^2 = 121$ sq. in.; so that $a = \sqrt{121} = 11$ inches.

**Example* iv. The area of the floor of a room is 210 square feet, and the area of the four walls is 609 square feet. If the height is 10 ft. 6 in., find the length and breadth.

Here $c = 10\frac{1}{2}$ ft., and it is required to find a and b.

Given the area of floor $= 210$ sq. ft.; $\quad ab = 210 \quad \ldots \ldots$ (i)

and area of four walls $= 609$ sq. ft.; $\quad 2c(a+b) = 609 \quad$ (ii).

Substituting in (ii) the value of c, we have

$$a + b = 29,$$

and

$$ab = 210.$$

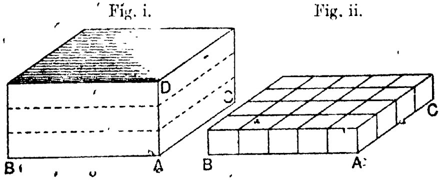
Hence $\quad a = 15$ feet, $b = 14$ feet.

83. *To find the* volume *of a rectangular solid, multiply together the number of linear units in the length, breadth, and height, and the result will be the number of* cubic *units in the volume.*

For example, if a rectangular solid is 5 feet long, 4 feet wide, and 3 feet high, the volume is $5 \times 4 \times 3 = 60$ cubic feet.

The reason of this may be thus explained.

Fig. i. Fig. ii.



Let ABCD be a rectangular solid, whose length AB is 5 feet, breadth AC 4 feet, and height AD 3 feet: then by reference to fig. i, it will be seen that the solid may be divided into three equal *slices*, each one foot thick. And each slice may be subdivided (as in fig. ii.) into cubical blocks, whose edges are one foot, that is, into cubic feet.

Now the number of cubic feet in one slice is 5×4: so that the number of cubic feet in the whole solid is $5 \times 4 \times 3$, or 60.

Thus if a rectangular solid measures a *units in length,* b *units in breadth, and* c *units in height, its volume contains* abc *cubic units.*

And if there are a *linear units in the edge of a cube, its volume contains* a × a × a, *or* a^3 *cubic units.*

These statements may be thus abridged:

the volume of a rectangular solid = length × breadth × height;
the volume of a cube = (edge)^3.

Example. A rectangular tank measures internally 8 feet in length, 6 feet in breadth, and 2 ft. 4 in. in depth: how many gallons will it hold, supposing 1 cubic foot contains $6\frac{1}{4}$ gallons?

$$\text{The } \textit{volume of the tank} = \textit{length} \times \textit{breadth} \times \textit{depth}$$
$$= 8 \times 6 \times 2\tfrac{1}{3} \text{ cubic feet.}$$

$$\therefore \text{ the } \textit{capacity} \text{ of the tank} = 8 \times 6 \times 2\tfrac{1}{3} \times 6\tfrac{1}{4} \text{ gallons}$$
$$= 700 \text{ gallons.}$$

84. *Given the volume of a rectangular solid and two of its dimensions, to find the third dimension.*

It was shewn in the last Article, that

the volume of a rectangular solid = length × breadth × height.

$$\text{Hence} \quad \textit{the height} = \frac{\textit{volume}}{\textit{length} \times \textit{breadth}} = \frac{\textit{volume}}{\textit{area of base}}.$$

Example i. Find the height of a rectangular solid whose volume is 7 c. ft. 864 c. in., length 4 feet, and breadth 1 ft. 3 in.

Here the volume = 7 c. ft. 864 c. in. = $7\frac{1}{2}$ c. ft.

$$\text{But} \quad \textit{height} = \frac{\textit{volume}}{\textit{length} \times \textit{breadth}} = \frac{7\frac{1}{2}}{4 \times 1\frac{1}{4}} \text{ feet} = 1 \text{ ft. } 6 \text{ in.}$$

Example ii. A rectangular tank holds $12\frac{1}{2}$ tons of water: if it is 32 feet long and 4 feet wide, find its depth; supposing 1 c. foot of water weighs 1000 oz.

Total weight of water = $12\frac{1}{2}$ tons = 448000 oz.

$$\therefore \text{ the volume of the tank} = 448000 \div 1000 \text{ c. ft.}$$
$$= 448 \text{ c. ft.}$$

$$\text{But} \quad \textit{depth} = \frac{\textit{volume}}{\textit{length} \times \textit{breadth}} = \frac{448}{32 \times 4} \text{ feet} = 3 \text{ ft. } 6 \text{ in.}$$

85. Suppose a closed box, which measures externally a inches long, b inches wide, and c inches high, is made of wood k inches thick.

(i) Then the internal dimensions are as follows:—

length $= a - 2k$; breadth $= b - 2k$; height $= c - 2k$.

Hence the *internal* surface may be found by the formula of Art. 82.

(ii) The number of cubic inches in the *capacity* of the box $= (a - 2k)(b - 2k)(c - 2k)$.

(iii) The number of cubic inches of *material* used in the construction of the box $= abc - (a - 2k)(b - 2k)(c - 2k)$.

**Example*. A zinc cistern (open at the top) measures externally 3 ft. 3 in. long, 2 ft. 3 in. broad, and 2ft. 1 in. deep, and its capacity is 75 gallons. If the bottom of the cistern is 1 inch thick, find the thickness of the sides. (Given 1 c. foot $= 6\frac{1}{4}$ gallons.)

Let x be the required thickness, expressed as the fraction of a foot. Then the *internal* dimensions are respectively $(3\frac{1}{4} - 2x)$ feet, $(2\frac{1}{4} - 2x)$ feet, and 2 feet.

Hence the capacity is $(3\frac{1}{4} - 2x)(2\frac{1}{4} - 2x) \times 2$ cubic feet,

or $$(3\tfrac{1}{4} - 2x)(2\tfrac{1}{4} - 2x)\, 2 \times 6\tfrac{1}{4} \text{ gallons.}$$

Hence $$(3\tfrac{1}{4} - 2x)(2\tfrac{1}{4} - 2x)\, 2 \times 6\tfrac{1}{4} = 75,$$

or, $$(13 - 8x)(9 - 8x) = 96.$$

Solving this quadratic, and selecting the positive root, we have $8x = 1$; hence $x = \frac{1}{8}$ foot $= 1\frac{1}{2}$ inch.

86. *Given the volume of a cube, to find the length of its edge.*

If there are a linear units in the edge of a cube, it has been shewn that its volume contains a^3 units of solid measure. Thus if V denotes the number of cubic units in the volume,

$$a^3 = V.$$
$$\therefore a = \sqrt[3]{V}.$$

Hence the number of linear units in the edge is found by taking the cube root *of the number of cubic units in the volume.*

Example. Find (to the nearest inch) the edge of a cubical cistern capable of containing 1000 gallons, supposing 1 gallon = 277·274 cubic inches.

Here $V = 277{\cdot}274 \times 1000$ c. in.

$= 277274$ c. in.

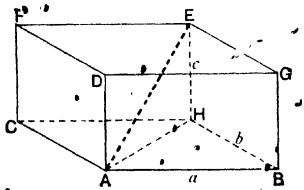
And $a = \sqrt[3]{V}$

$= \sqrt[3]{277274}$ in.

$= 65$ in. nearly.

		277 274·000	65·2
		216	
$(60)^2 \times 3 =$	10800	61274	
$(60) \times 5 \times 3 =$	900		
$5^2 =$	25		
	11725	58625	
$(650)^2 \times 3 =$	1267500	2649000	

***87.** *To find the diagonal of a rectangular solid when the dimensions are given.*



Let AGF be a rectangular solid, whose length, breadth, and height are respectively a, b and c units.

It is required to find the length of the diagonal AE.

The $\triangle$ ABH is right-angled at B,

$$\therefore \text{AH}^2 = \text{AB}^2 + \text{BH}^2 = a^2 + b^2 \ldots\ldots\ldots\ldots \text{(i)},$$

and the $\triangle$ AHE is right-angled at H,

$$\therefore \text{AE}^2 = \text{AH}^2 + \text{EH}^2 = a^2 + b^2 + c^2 \ldots\ldots\ldots\ldots \text{(ii)},$$

$$\therefore \text{ the diagonal AE} = \sqrt{a^2 + b^2 + c^2}.$$

NOTE. In a cube the length, breadth and height are all equal; hence if the length of each edge is a units,

$$\text{the diagonal} = \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = a\sqrt{3}.$$

EXAMPLES. XIII. A.

(1) Rectangular Solids.

[Elementary Course.]

(Surfaces.)

1. Find the number of square feet in the surface of a rectangular solid whose length is 20 feet, breadth 15 feet, height 10 feet.

2. Find in square feet the surface of a cube whose edges measure 4 feet.

3. Find the surface of a rectangular solid whose length is 12 feet, breadth 5 ft. 4 in., height 5 ft. 3 in.

4. Find in square feet the surface of a cube whose edges measure 4 ft. 6 in.

5. Find the cost of painting the outside of a chest whose length is 7 ft. 6 in., breadth 6 ft. 8 in., and height 6 feet, at the rate of $4d.$ a square yard.

6. Find the surface (in square yards and square feet) of the four walls of a room 20 feet long, 12 feet wide, 10 feet high.

7. Find the cost of papering the four walls of a room whose length is 20 ft. 6 in., breadth 15 ft. 6 in., and height 11 ft. 3 in., at $8d.$ a square yard.

8. A room 20 feet long, 16 feet wide, and 10 feet high, contains a door 8 feet high and 4 feet wide, and two windows each measuring 5 feet by 3 feet: what area of paper will be required to cover the walls?

9. How many yards of paper 2 ft. 4 in. wide are required for the walls of a room 23 ft. 3 in. long, 18 ft. 9 in. wide, and 14 feet high?

10. Find the cost of papering a room 17 ft. 4 in. long, 12 feet 8 in. broad, and 12 feet high, with paper 2 feet wide at $7\frac{1}{2}d.$ a yard.

11. It costs $19s.\ 9\frac{1}{2}d.$ to paper the walls of a room 19 ft. 4 in. long, 10 ft. 8 in. wide, and 9 ft. 6 in. high. If the paper is 2 feet wide, what is its price per yard?

12. A cubical chest, each edge of which measures externally 5 feet 5 in., is made of deal one inch thick. What are its inner dimensions? Find its inner surface in square feet.

13. A chest whose external length, breadth, and height are respectively 5 ft. 9 in., 4 ft. 3 in., and 3 ft. 3 in., is made of deal $1\frac{1}{2}$ inches thick. What are its inner dimensions? Find the cost of lining it with thin metal at the rate of 8*d.* per square foot.

14. The external dimensions of a chest are 4 ft. 6 in., 3 ft. 8 in. and 4 ft. 2 in.: if the wood of which it is made is one inch thick, find the cost of painting it *inside* at the rate of 8*d.* per dozen square feet.

15. A room 20 feet long by 16 broad is to be panelled to a height of 6 feet, allowing 5 feet for the width of a door and 4 ft. 6 in. for a fireplace. How many panels a yard long and 15 inches wide will be required?

16. What is the length of the edge of a cube whose surface is

(i) 9 sq. ft. 54 sq. in.,

(ii) 15 sq. yds. 0 sq. ft. 54 sq. in.

17. It costs 1*s.* $7\frac{1}{2}$*d.* to paint the surface of a cube at the rate of 3*s.* for 13 square feet. Find the length of each edge.

(*Volumes.*)

(*Given the dimensions, to find the volume.*)

18. Find the volume of the rectangular solids in which

(i) the length is 8 feet, the breadth 6 feet, and the height 5 feet,

(ii) the length is 6 ft. 8 in., the breadth 5 ft. 3 in., and the height 2 feet.

19. Find the volume of the cubes in which each edge measures

(i) 6 ft. 6 in., (ii) 3 yds. 2 ft. 3 in.

20. Find the value of a rectangular block of metal whose dimensions are 8 feet, 5 feet and 3 feet, at the rate of 17*s.* 6*d.* per cubic foot.

21. How many gallons are contained in a cubical vessel which measures internally 4 feet in length, breadth and depth, supposing 1 cubic foot to contain $6\frac{1}{4}$ gallons?

22. A rectangular tank is 16 feet long, 8 feet wide, and 7 feet deep: how many tons of water will it hold? [1 cubic foot of water weighs 1000 oz.]

23. How long will it take to dig a trench 160 yards long, 16 feet wide, and 14 feet deep, if 30 tons of earth are removed in a day? [1 cubic foot of earth weighs $92\frac{1}{4}$ lbs.]

24. A brick (with mortar) occupies a space 9 inches long, $4\frac{1}{2}$ inches broad, and 3 inches high. How many bricks will be required for a wall 30 yards long, 6 feet high, and $13\frac{1}{2}$ inches thick?

25. The dimensions of a cistern are—length 5 ft. 4 in., breadth 4 ft. 6 in., depth 1 yard. How many gallons of water is it capable of containing? [1 cubic foot contains $6\frac{1}{4}$ gallons.]

(Given the volume and two dimensions, to find the third.)

26. Find the height of the rectangular solid in which

(i) the volume is 792 cubic inches, the length 11 inches, and the breadth 9 inches,

(ii) the volume is 3 c. ft. 1296 c. in., the length 2 feet, and the breadth 1 ft. 6 in.,

(iii) the volume is 50 cubic feet, and the area of the base 9 sq. ft. 54 sq. in.

27. What must be the depth of a tank whose base is a square on a side of 1 yard, if it holds as much water as a second tank of which the dimensions are 4 ft. 6 in., 2 ft. 3 in., and 1 ft. 4 in.?

28 A rectangular block of metal whose dimensions are 1 ft. 6 in., 1 foot, and 10 inches, is thrown into a cistern partly full of water: if the cistern stands on a base 2 ft. 6 in. by 1 ft. 4 in. and the block is completely immersed, how high will it cause the surface of the water to rise?

29. A tank, the area of whose base is $9\frac{1}{4}$ square feet, is capable of containing 60 gallons of water: find its depth to the nearest inch. [1 gallon $= 277\frac{1}{4}$ cubic inches nearly.]

EXAMPLES. XIII. B.

ON RECTANGULAR SOLIDS.

[*Higher Course.*]

(*Surfaces.*)

1. Find the edge of a cube having a surface equal to the sum of the surfaces of two cubes whose edges are 120 inches and 209 inches.

2. The sides of a box are $\frac{1}{2}$ inch thick, and the lid and bottom are one inch thick. If the outer dimensions are:—length 4 ft. 7 in., breadth 3 ft. 5 in., height 3 ft. 2 in., find the interior surface in square feet.

3. A cubical chest, measuring externally 2 ft. 7 in. along each edge, is built of wood of uniform thickness. If its total inner surface is $37\frac{1}{2}$ sq. ft., what is the thickness of the wood?

4. A closed box is made of wood of uniform thickness. Its external dimensions are 11 inches, 9 inches and 7 inches, and its *inner* surface is 286 square inches; find the thickness of the wood.

5. A room is $12\frac{1}{2}$ feet high, and it is half as long again as it is broad. If the area of the four walls is 875 square feet, find the length and breadth.

6. A room is 20 ft. 3 in. long by 15 ft. 9 in. wide: if the cost of painting its four walls at the rate of 3*d.* per square yard is £1. 7*s.*, find the height of the room.

7. The dimensions of a rectangular solid are proportional to 3, 4, and 5. If the whole surface contains 2350 square inches, find the length, breadth, and height.

8. The surface of a rectangular solid is 1000 square inches; if its length and breadth are respectively 1 ft. 3 in. and 1 ft. 2 in., find its height.

9. The whole surface of a rectangular solid contains 1224 square feet, and the four vertical faces together contain 744 square feet. If the height is 12 feet, find the length and breadth.

10. A rectangular solid stands on a square base, and its whole surface contains 1292 square feet; if the height is 10 ft. 6 in., find the remaining dimensions.

11. The whole surface of a rectangular solid contains 724 square feet; the area of the base is 132 square feet, and one of the vertical faces contains 110 square feet; find the length, breadth, and height.

(*Volumes.*)

NOTE. *In the following examples suppose* 1 *cubic foot of water weighs* 1000 *oz.;* 1 *gallon* = 277·25 *cubic inches.*

12. What must be the depth of a tank whose internal length and breadth are 5 ft. 4 in. and 5 ft. 3 in., in order that it may hold a ton of water?

13. A cistern with vertical sides stands upon a rectangular base measuring internally 6 ft. 8 in. long, 3 ft. 9 in. broad, and 2 feet deep: find to the nearest second in what time it will be filled by a pipe which admits 10 gallons a minute.

14. A thousand gallons are poured into a cistern whose base is a square, filling it to a depth of 18 inches. Find (to the nearest inch) the length and breadth of the cistern.

15. Water flows into a rectangular tank through a pipe which admits 15 gallons per minute. Find approximately at what rate (in inches per hour) the water will rise in the tank, if the dimensions of its base are 24 feet and 18 feet.

16. A closed box, whose external dimensions are 4 ft. 8 in., 4 ft. 2 in, and 2 ft. 6 in., is made of deal one inch thick. Find the weight of the box given that 1 cubic foot of deal weighs 912 ounces.

17. A rectangular zinc cistern (open at the top) measures externally 2 ft. 8 in. long, 1 ft. 9 in. broad, and 1 ft. $4\frac{1}{2}$ in. deep. If the metal is $\frac{1}{2}$ inch thick, find the weight of the cistern (to the nearest pound), and find also its total weight when filled with water. [1 cubic foot of zinc weighs 7215 oz.]

18. A cubical block of ice, each edge of which is 3 feet, is placed in an empty tank, and when the ice has melted it is found that the water is 15 inches deep. If the tank stands upon a square base, find (to the nearest hundredth of an inch) its length and breadth. [N.B. The volume of ice is greater than the volume of the same weight of water; suppose 1 cubic foot of ice weighs 930 oz. and 1 cubic foot of water weighs 1000 oz.]

Questions involving the extraction of cube roots.

19. Find the edge of a cube equal in volume to a rectangular solid whose dimensions are 6 ft. 3 in., 2 ft. 6 in., and 1 foot.

20. Find (in square feet and inches) the surface of a cube whose volume is 4 c. ft. 1088 c. in.

21. The edges of a rectangular block of granite are proportional to 2, 3, and 5, and its volume is 101 c. ft. 432 c. in. Find its dimensions.

22. Find (to the nearest tenth of an inch) the dimensions of a cubical cistern capable of containing 800 gallons.

23. Find (to the nearest hundredth of an inch) the edge of a cubical block of lead weighing one ton, having given that 1 c. foot of lead weighs 709½ lbs.

24. Find the edge of a cubical mass of brass, whose weight is equal to that of a rectangular block of iron measuring 2 ft. 3 in. long, 1 ft. 4 in. broad, and 12 inches thick. [1 cubic foot of brass weighs 8000 ounces, 1 c. foot of iron weighs 7788 ounces.]

(*Diagonals.*)

25. Find the surface and volume of a cube, in which the diagonal *of each face* is 1 ft. 3 in.

26. The volume of a cube is 4 c. ft. 1088 c. in.; find the length of its diagonal.

27. Find the surface and volume of a cube whose diagonal is 2 ft. 6 in.

28. Find the length and breadth of a rectangular solid, having given that the diagonal is 13 inches, the height 3 inches, and the area of the base is 48 square inches.

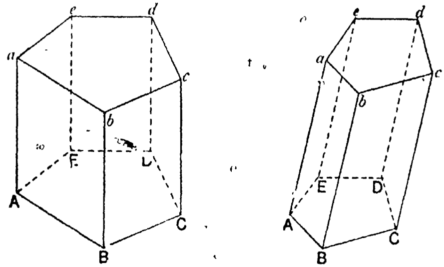
29. The diagonal of a rectangular solid is 29 inches, and its volume is 4032 cubic inches; if the thickness is one foot, find the length and breadth.

30. The diagonal of a rectangular solid is 37 inches, and the whole surface is 2352 square inches; shew that the sum of the edges is 61 inches.

CHAPTER XIV.

THE PRISM.

88. Definition. A **prism** is a solid bounded by plane faces of which two, called the *ends*, are similar, equal, and parallel figures, and the other faces are parallelograms.



Thus in the prisms represented in the diagram, the two faces ABCDE, $abcde$ are the *ends*; the remaining faces, such as ABba, BCcb, &c., are seen to be *parallelograms*. The ends of the prisms here represented are pentagons, but a prism may have for its ends any rectilineal figure whatever. Either end, on which the prism may be supposed to stand, is called the *base*.

The prism is *right*, when the edges which join corresponding vertices of the ends are perpendicular to the ends: if otherwise the prism is *oblique*. Thus if Aa, Bb, Cc, &c. are perpendicular to the ends the prism is right, and the faces ABba, BCcb, &c. are rectangles. It will be seen that the edges Aa, Bb, Cc, &c. are all equal.

89. *To find the surface and volume of a right prism.*

In the above figure let AB $= a$ units of length, BC $= b$, CD $= c$, &c.; and let the height A$a = h$. Let the area of the base contain E square units; and let the perimeter of the base $= p$.

Then the area of the rectangle ABba $= ah$ square units; and the area of the remaining rectangles are respectively bh, ch, &c.

Hence (i) *the lateral surface of prism*

$$= ah + bh + ch + \ldots$$

$$= (a+b+c+\ldots)\, h \text{ square units.}$$

$$= (\textit{perimeter of base}) \times \textit{height}.$$

The *whole surface* of prism = lateral surface + area of the ends

$$= ph + 2\mathsf{E}.$$

(ii) the *volume of prism* = (*area of base*) × *height*

$$= \mathsf{E}h \text{ cubic units.}$$

Thus if V denotes the volume of the prism, we have

$$\mathsf{V} = \mathsf{E}h.$$

Hence $$h = \frac{\mathsf{V}}{\mathsf{E}}; \text{ and } \mathsf{E} = \frac{\mathsf{V}}{h}.$$

That is to say,

$$\textit{height of prism} = \frac{\textit{volume}}{\textit{area of base}}; \quad \textit{area of base} = \frac{\textit{volume}}{\textit{height}}.$$

Example i. The base of a right prism is a triangle whose sides are 1 ft. 9 in., 1 ft. 8 in., and 1 ft. 1 in.; if the height is 8 feet, find the whole surface and volume.

Here $a = 21$ inches, $b = 20$ inches, $c = 13$ inches.

∴ the perimeter of the base = 54 inches = $4\frac{1}{2}$ feet.

(i) Now *lateral surface of prism* = (*perimeter of base*) × *height*

$$= 4{\cdot}5 \times 8 \text{ sq. ft.} = 36 \text{ sq. ft.}$$

Again, the area of the base $= \sqrt{s\,.\,(s-a)(s-b)(s-c)}$ Art. 23.

$$= \sqrt{27 \times 6 \times 7 \times 14} \text{ sq. in.}$$

$$= 126 \text{ sq. in.} = \frac{7}{8} \text{ sq. ft.}$$

Hence the *whole* surface = lateral surface + area of two ends

$$= 36 + \frac{7}{4} \text{ sq. ft.} = 37\tfrac{3}{4} \text{ sq. ft.}$$

(ii) The *volume of prism* = (*area of base*) × *height*

$$= \frac{7}{8} \text{ sq. ft.} \times 8 \text{ ft.}$$

$$= 7 \text{ cubic feet.}$$

Example ii. The ends of a granite column are regular hexagons on a side of 5 inches. If the vertical surface is 13 sq. ft. 108 sq. in., find the height.

$$\text{Here the perimeter} = 5 \times 6 \text{ inches} = 2{\cdot}5 \text{ feet.}$$

$$\text{And } \textit{vertical surface} = (\textit{perimeter of base}) \times \textit{height}$$

that is, $$13\tfrac{3}{4} \text{ sq. ft.} = \frac{5}{2} \times \text{height}$$

Hence, $$\text{the height} = 13\tfrac{3}{4} \div \frac{5}{2} = 5 \text{ ft. } 6 \text{ in.}$$

Example iii. Water flows at the rate of 6 inches per second through a wooden pipe, whose cross-section is an equilateral triangle. If 6 tons of water are passed through the pipe in 20 minutes, find the dimensions of its section. [1 c. foot of water weighs 1000 oz.]

Let each side of the equilateral section be $2m$ feet; then its

$$\text{area} = m^2\sqrt{3} \text{ sq. ft.} \qquad \text{Art. 25.}$$

In 20 minutes, at the rate of 6 inches per second, a column of water 600 feet in length will have passed;

$$\therefore \text{ the column of water passed} = m^2\sqrt{3} \times 600 \text{ c. ft.}$$

$$\text{And 6 tons of water} = \frac{6 \times 20 \times 112 \times 16}{1000} \text{ c. ft.} = 215{\cdot}04 \text{ c. ft}$$

$$\therefore m^2\sqrt{3} \times 600 = 215{\cdot}04,$$

$$\therefore m^2 = \frac{215{\cdot}04}{\sqrt{3} \times 600} = {\cdot}2069\ldots \text{ sq. ft,} \quad m = \sqrt{{\cdot}2069} = {\cdot}45 \text{ ft.}$$

$$\therefore \text{ each side of section, or } 2m, = {\cdot}9 \text{ ft.} = 10{\cdot}8 \text{ inches.}$$

90. A **parallelepiped** is a solid bounded by six faces, which are all parallelograms, opposite faces being parallel and equal.

Fig. i.

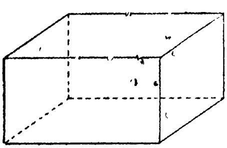
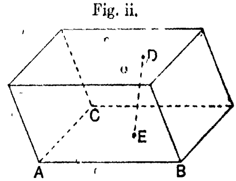


Fig. ii.



It will be seen from the figures that a parallelepiped is a prism of which the *ends* (as well as the lateral faces) are parallelograms.

A parallelepiped may be *rectangular*, as in fig. i, or *oblique* as in fig. ii. Methods of finding the surface and volume of a rectangular parallelepiped (or rectangular solid) have already been given.

*91. *Prisms on the same base, and between the same parallel planes have equal volumes.*

That is to say, the volume of an oblique prism is equal to that of a right prism on the same base and of the same *height* (measured at right angles to the base):
or, *the volume of an oblique prism = (area of base) × height.*

NOTE. It may here be noticed, in anticipation of what follows, that a similar theorem holds good of *cylinders, pyramids,* and *cones.* For example:—The volume of an oblique pyramid is equal to that of a right pyramid on the same base and of the same perpendicular height.

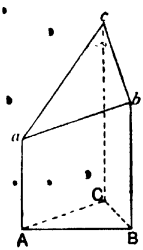
Example. Find the volume of an oblique parallelepiped, whose perpendicular height is h feet, the sides of the base being a feet and b feet, inclined at an angle A°.

In fig. ii, let $AB = a$, $AC = b$, and the perp. height $DE = h$.

Then the area of the base $= ab \sin CAB = ab \sin A$. Art. 31.

And *volume of parallelepiped = (area of base) × perp. height.*
$= hab \sin A$.

*92. The accompanying figure represents a solid formed by cutting a right triangular prism by a plane abc, which is not parallel to the base ABC.



The volume of such a figure (called the *oblique frustum* of a triangular prism) is given by the following formula.

Volume = (area of base)
× (*the mean of the lateral edges*)

$= (\textit{area of base}) \times \frac{1}{3}(Aa + Bb + Cc).$

EXAMPLES. XIV.

ON PRISMS.

[*Elementary Course.*]

(*Surfaces.*)

1. The base of a right prism, 1 ft. 8 in. high, is an equilateral triangle on a side of 1 foot; find the area of its rectangular faces.

2. The base of a right prism, 2 ft. 3 in. high, is a regular pentagon whose side is 1 ft. 4 in.; find the area of its lateral surface.

3. Find the cost of polishing the vertical surface of a granite column, 10 feet high, standing on an octagonal base whose side is 8 inches, at the rate of 1*s.* $1\frac{1}{2}d.$ per square foot.

(*Surfaces and Volumes.*)

4. The base of a right prism is a right-angled triangle, whose sides containing the right angle are 3 inches and 4 inches. If the height of the prism is 8 inches, find the volume and the whole surface.

5. A right prism stands upon a triangular base, whose sides are 13, 14, and 15 inches. If the height is 10 inches, find its volume and whole surface.

6. Find the weight of a right prism of silver 5 inches long, the ends being triangles whose sides are 2·6 inches, 2·5 inches, and 1·7 inches. [Given one cubic inch of silver weighs 6·08 ounces.]

7. A right prism, 5 inches high, stands upon a quadrilateral base ABCD. Find its volume and whole surface, if AB = 7 inches, BC = 5 inches, CD = 4 inches, DA = 4 inches, and the angles at A and D are right angles.

8. The base of a right prism is a trapezium whose parallel sides are 11 inches and 7 inches respectively, the distance between them being 6 inches: if the height of the prism is 1 foot, find the volume in cubic feet.

9. A trench measures 13 feet across the top, and 11 feet across the bottom, and it has a uniform depth of 5 feet. If its length is 40 feet, how many gallons of water will it hold? [Given 1 cubic foot = $6\frac{1}{4}$ gallons.]

10. A trench has a uniform depth of 7 feet, and its width across the top is 8 feet greater than across the bottom. If its width across the top is 20 feet, and its length 16 yards, how many tons of water will it hold? [Given 1 cubic foot of water weighs 1000 oz.]

11. The cost of polishing the vertical surface of a granite column is £5. 4*s.*, this being at the rate of 1*s.* 4*d.* per square foot. If the cross-section of the column is a regular nonagon on a side of 8 inches, find its height.

12. The volume of a prism standing on a triangular base is 3 c. ft. 1116 c. in. If the sides of the base are 3 ft. 1 in., 2 ft. 11 in., and 1 foot, find the height of the prism.

13. The weight of a brass prism standing on a triangular base is 875 lbs. If the sides of the base are 25 inches, 24 inches, and 7 inches, find the height of the prism, supposing that 1 cubic foot of brass weighs 8000 ounces.

[*Higher Course.*]

(*In the following Examples suppose* 1 *cubic foot of water weighs* 1000 *oz.*, 1 *gallon* = 277·25 *cubic inches.*)

14. The cross-section of a right prism is a quadrilateral figure ABCD, in which

AB = 7 inches, BC = 20 inches, CD = 15 inches, DA = 24 inches,

and the angles at A and C are right angles. If the height of the prism is 18 inches, find its whole surface and volume.

15. The cross-section of a right prism is a quadrilateral figure ABCD, in which

AB = 9 inches, BC = 14 inches, CD = 13 inches, DA = 12 inches,

and the angle at A is a right angle. If the volume is 2070 cubic inches, find the height and the surface.

16. A railway cutting, 140 yards in length, has a uniform depth of 12 feet, and its dimensions across the top and bottom are respectively 84 feet and 44 feet. Find the cost of excavation at the rate of half-a-crown per ton of earth removed, having given one cubic foot of earth weighs $92\frac{1}{2}$ lbs.

17. The whole surface of a right prism contains $9\frac{1}{2}$ square feet, the ends being triangles whose sides are 4 ft. 3 in., 3 ft. 1 in., and 1 ft. 8 in.: find the height.

18. The surface of a right prism contains 720 square inches, and its transverse section is a triangle whose sides are 2 ft. 1 in., 1 ft. 5 in. and 1 foot: find the volume.

19. A granite pillar whose ends are regular hexagons on a side of 1 ft. 2 in. weighs 7 tons. Find its height (to the nearest tenth of a foot) supposing that granite is 2·7 times as heavy as water. [$\sqrt{3} = 1\cdot73205$.]

20. Water flows at the rate of 30 yards per minute through a wooden pipe, whose cross-section is a square on a side of 4 inches. How long would it take to fill a cubical cistern, whose internal edge is 6 feet?

21. Eighteen tons of water flowing through a wooden pipe at the rate of $3\frac{1}{2}$ feet per second, are passed into a tank in half-an-hour. If the cross-section of the pipe is a square, find its dimensions.

22. The ends of a wooden trough (4 feet in length) are equilateral triangles, whose upper sides are horizontal. If 9 gallons of water are poured into it, find (to the nearest tenth of an inch) how high the surface of the water will stand above the lower edge of the trough.

23. The transverse section of a block of timber is a square on a side of one foot, and its length is 5 feet. If the longitudinal edges are planed down so that the transverse section becomes throughout a regular octagon, find the volume of the remainder. What decimal part of the whole block has been cut away?

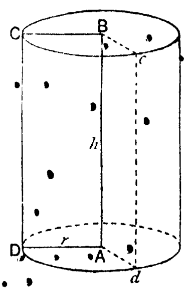
24. The base of a prism is a right-angled triangle whose hypotenuse is 17 inches. If the height is 1 foot, and the sum of the rectangular faces 480 square inches, find the other sides of the base.

CHAPTER XV.

THE RIGHT CIRCULAR CYLINDER.

93. Definition. A **right circular cylinder** is a solid described by the revolution of a rectangle about one of its sides which remains fixed.

Thus if the rectangle ABCD revolves about the side AB, it describes the cylinder represented in the figure. AB is said to be the *axis* of the cylinder; and the height of a right cylinder is the length of its axis.



The whole surface of the cylinder consists of the *curved surface*, described by CD, and the two *circular ends* described by AD and BC. Either end, on which the cylinder may be supposed to rest, may be called its *base*.

A resemblance in character will be observed between the cylinder and prism: indeed a cylinder may be roughly described as a prism whose ends are circular. Accordingly, the rules for finding the surface and volume of a cylinder will be found to correspond with those already given for the prism.

Note. The above definition refers only to a *right circular cylinder*. Cylinders may exist whose axes are not perpendicular to the ends, and whose ends are not circular. Such cylinders however are beyond the scope of the present text-book.

94. *To find the surface and volume of a cylinder.*

In the figure of the preceding page let the height $AB = h$ units of length, and let the radius AC of the base $= r$.

(i) *The curved surface of cylinder*

$$= (\textit{circumference of base}) \times \textit{height}$$
$$= 2\pi r \times h$$
$$= 2\pi rh \text{ square units.}$$

The *whole surface* of cylinder = curved surface + area of ends

$$= 2\pi rh + 2\pi r^2$$
$$= 2\pi r(h + r).$$

(ii) *The volume of cylinder* = (*area of base*) $\times$ *height*

$$= \pi r^2 \times h$$
$$= \pi r^2 h \text{ cubic units.}$$

Hence if V denotes the volume of a cylinder, we have

$$V = \pi r^2 h.$$

From which $h = \dfrac{V}{\pi r^2}$, and $r = \sqrt{\dfrac{V}{\pi h}}$.

Fig. i.

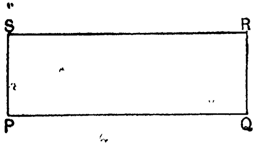
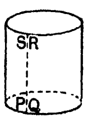


Fig. ii.



NOTE. The formula which gives the curved surface of a cylinder may be illustrated by bending round a rectangular piece of paper PQRS until the lines PS and QR coincide. In this way the paper may be made to take the shape of the curved surface of a cylinder; PQ, the length of the paper, becoming the circumference of the base of the cylinder.

Thus *the curved surface of cylinder*

= the area of the rectangular paper

$= PQ \times PS$

$=$ (*circumference of cylinder*) $\times$ *height.*

Example i. Find the curved surface and volume of a cylinder whose height is 1 ft. 2 in., and the diameter of whose base is 1 yard.

Here $h = \frac{7}{6}$ ft.; $r = \frac{3}{2}$ ft.

(i) *Curved surface* $= 2\pi rh$

$$= 2 \times \frac{22}{7} \times \frac{3}{2} \times \frac{7}{6} \text{ sq. ft.} = 11 \text{ sq. ft.}$$

(ii) *Volume* $= \pi r^2 h$

$$= \frac{22}{7} \times \left(\frac{3}{2}\right)^2 \times \frac{7}{6} \text{ c. ft.} = 8\tfrac{1}{4} \text{ c. ft.}$$

Example ii. The weight of a cylindrical granite column is 8 tons 5 cwt., and its height is 8 ft. 9 in. Find its diameter, supposing 1 cubic foot of granite weighs 168 lbs.

The weight of column $= 8\frac{1}{4}$ tons $= 8\frac{1}{4} \times 20 \times 112$ lbs.;

$$\therefore \text{ the volume of column} = \frac{8\frac{1}{4} \times 20 \times 112}{168} = 110 \text{ c. ft.}$$

Hence $\pi r^2 h = 110,$

$$\therefore r^2 = \frac{110}{\pi h} = \frac{110}{\frac{22}{7} \times 8\frac{3}{4}} = 4.$$

$$\therefore r = 2 \text{ feet.}$$

That is, the diameter of the cylinder is 4 feet.

Example iii. From a cylindrical tank $4\frac{1}{2}$ feet in diameter water is drawn off at the rate of 110 gallons per hour. Find (to the tenth of an inch) by how much the surface would be lowered in 27 minutes. ($\pi = 3{\cdot}1416$; and 1 gallon $= 277{\cdot}25$ cubic inches).

Here $r = 27$ inches; suppose the surface is lowered by h inches. Then the cylindrical mass of water drawn off $= \pi r^2 h$ c. in.;

and this by hypothesis is equal to $\frac{27}{60} \times 110 \times 277{\cdot}25$ c. in.,

$$\therefore \pi (27)^2 h = \frac{27}{60} \times 110 \times 277{\cdot}25,$$

Hence $h = \dfrac{11 \times 277{\cdot}25}{6 \times 27 \times \pi}$

$= 6$ inches nearly.

$$\begin{array}{r|r|l}
 & 277{\cdot}25 & \\
 & 11 & \\
6 & 3049{\cdot}75 & \\
27\begin{cases}9\\3\end{cases} & 508{\cdot}2916 & \\
 & 56{\cdot}4768 & \\
3{\cdot}1416 & 18{\cdot}8256 & 5{\cdot}99\ldots \\
 & 15\ 7080 & \\
 & 3\ 1176 & \\
 & 2\ 8274 & \\
 & 2902 & \\
 & 2827 &
\end{array}$$

95. The surface and volume of a hollow cylinder (open at the ends) are thus found.

Let the height $= h$, the external radius $= r$, and the thickness $= k$; so that the internal radius $= r - k$.

(i) Then the *external* curved surface $= 2\pi rh$

the *internal* curved surface $= 2\pi (r - k) h$,

and *each* of the two circular rings, which form the ends

$$\pi r^2 - \pi (r - k)^2. \quad \text{Art. 42.}$$

The whole surface is found by adding these results.

(ii) The volume of material in a hollow cylinder is the difference between the volume of a cylinder having the given external dimensions, and the volume of a cylinder having the internal dimensions. Hence the required volume is

$$\pi r^2 h - \pi (r - k)^2 h = \pi h \{r^2 - (r - k)^2\}.$$

Example (i). Find the whole surface of a hollow cylinder, open at the ends, if the length is 8 inches, the external diameter 10 inches, and the thickness 2 inches ($\pi = 3{\cdot}1416$)

Here $h = 8$ in.; the external radius $= 5$ in., and the internal radius $= 3$ in.

External curved surface $= 2\pi \times 5 \times 8 = 80\pi$ sq. in.

Internal curved surface $= 2\pi \times 3 \times 8 = 48\pi$ sq. in.

The two circular rings $= 2\pi (5^2 - 3^2) = 32\pi$ sq. in.

Hence the whole surface $= \pi (80 + 48 + 32)$ sq. in.

$= 160 \times \pi$ sq. in $= 502{\cdot}65$ sq. in.

[Obs. A point to be noticed in this and the following example is that numerical work is saved by not substituting the value of π until the last stage of the work is reached.]

Example ii. Find the weight of a cylindrical iron pipe 64 feet long, the external diameter being $2\frac{1}{4}$ inches, and the thickness $\frac{1}{8}$ inch; supposing 1 cubic foot of iron weighs 486 lbs. ($\pi = 3{\cdot}1416$).

Here $h = 64$ ft. $= 768$ inches; external radius $= 1\frac{1}{8}$ inch; internal radius $= 1$ inch.

Hence volume of material in pipe $= 768\pi \left\{\left(\frac{9}{8}\right)^2 - 1\right\}$ c. in.

$$= 768\pi \times \frac{17}{8} \times \frac{1}{8} \text{ c. in.},$$

$$\therefore \text{ weight} = 768\pi \times \frac{17}{8} \times \frac{1}{8} \times \frac{486}{1728} \text{ lbs}$$

$$= \frac{\pi \times 17 \times 27}{8} \text{ lbs.} = 180\tfrac{1}{4} \text{ lbs. nearly.}$$

```
    3 1416
       17
   31 416
   21 991
   53 40|7
        9
  480 66|3
        3
8|1441 98|9
  180 25  nearly
```

**Example* iii. In a hollow cylinder, open at both ends, there are 678·6 cubic inches of material. if the length is 9 inches, and the external diameter 14 inches, find the thickness. ($\pi = 3{\cdot}1416$.)

Here $h = 9$in., $r = 7$in.; let the thickness $= k$ inches.

$$\therefore 9\pi\,\{7^2 - (7-k)^2\} = 678{\cdot}6,$$

$$7^2 - (7-k)^2 = \frac{678{\cdot}6}{\pi \times 9},$$

$$49 - (7-k)^2 = 24 \text{ nearly},$$

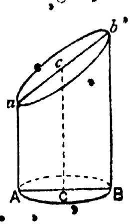
$$25 = (7-k)^2,$$

$$\therefore 5 = 7 - k,$$

$$k = 2 \text{ inches}$$

```
      9 | 678 6
 3 1416 |  75 4   |24 000
        |  62 832
        |  12 568
        |  12 566
```

***96.** The solid represented in the annexed figure is formed by cutting a right circular cylinder by a plane not parallel to the base. Such a solid is sometimes called the *oblique frustum* of a cylinder.



The curved surface and volume of this solid are equal to those of a right cylinder having the same base and the *mean height* Cc.

Hence

(i) *curved surface* $= 2\pi r \,.\, Cc = 2\pi r \times \dfrac{h_1 + h_2}{2}$,

(ii) *volume* $= \pi r^2 \,.\, Cc = \pi r^2 \times \dfrac{h_1 + h_2}{2}$,

where h_1 and h_2 denote the greatest and least heights.

EXAMPLES. XV. A.

ON CYLINDERS.

[*Elementary Course.*]

(*In the following Examples take* $\pi = \frac{22}{7}$.)

(*Surfaces.*)

1. Find the *curved* surfaces of the cylinders in which

(i) the radius of the base is 7 inches, and the height 5 inches.

(ii) the diameter of the base is 3 ft. 6 in. and the height 3 ft. 4 in.

2. Find the *whole* surface of the cylinders in which

(i) the radius of the base is $3\frac{1}{2}$ inches, and the height 8 inches.

(ii) the diameter of the base is 2 yds. 2 ft. 2 in., and the height 1 yd. 1 ft.

3. The radius of the base of a cylinder is 5 inches, and its curved surface is 440 square inches; find its height.

4. The diameter of a cylindrical granite column is 2 feet, and its height is 14 feet; find the cost of polishing its curved surface at the rate of 1*s.* 6*d.* per square foot.

5. How many square yards are covered in 100 revolutions of a cylindrical roller whose length is 4 ft. 6 in., and diameter 3 ft. 6 in.?

(*Volumes.*)

6. Find the volumes of the cylinders in which

(i) the radius of the base is 7 inches, and the height 5 inches;

(ii) the diameter of the base is 3 ft. 6 in., and the height 3 ft. 4 in.

7. Find (in pounds) the weight of a solid iron cylinder 1 ft. 9 in. long, the diameter of the base being 2 feet; supposing that 1 cubic foot of iron weighs $486\frac{3}{4}$ lbs.

8. How many gallons are contained in a cylindrical tank whose depth is 2 ft. 4 in., and whose base is $2\frac{2}{5}$ feet in diameter: supposing that 1 cubic foot $= 6\frac{1}{4}$ gallons?

9. Find the cost of boring a semi-cylindrical tunnel 30 feet in diameter and 88 yards in length, at the rate of 15*s*. 9*d*. for every cubic yard excavated.

10. Find the heights of the cylinders in which

(i) the volume is 308 cubic inches, and the radius of the base 3·5 inches;

(ii) the volume is 24 c. ft. 1296 c. in., and the diameter of the base 1 yard.

11. A cylindrical tank contains 4400 gallons: find its depth, if the diameter of its base is 5 ft. 4 in. Suppose 1 cubic foot $= 6\frac{1}{4}$ gallons.

12. It costs £343. 15*s*. to bore a cylindrical shaft, at the rate of 15*s*. 9*d*. for every cubic yard of rock removed. If the diameter of the shaft is 5 ft., what is its depth?

13. The volume of a cylinder 14 inches long is equal to that of a cube having an edge of 11 inches. Find the radius of the cylinder.

14. What is the internal diameter of a cylindrical tank 8 feet deep, if it is capable of containing $5\frac{1}{2}$ tons of water? (1 cubic foot of water weighs 1000 oz.)

(*Hollow cylinders.*)

15. Find the surface and volume of a hollow cylinder (open at both ends) whose *external* diameter is 44 inches, thickness 2 inches, and height 25 inches.

16. Find (to the nearest lb.) the weight of 14 feet of lead piping, whose *internal* diameter is 1 inch and thickness $\frac{1}{4}$ inch. Given 1 cubic inch of lead weighs $6\frac{1}{2}$ oz.

17. An iron roller is in the shape of a hollow cylinder whose length is 4 feet, external diameter 2 ft. 8 in., and thickness 4 inches. Find (in pounds) its weight, supposing one cubic foot of iron to weigh 486 lbs.

EXAMPLES. XV. B.

ON THE CYLINDER.

[Higher Course.]

(In the following Examples take $\pi = \frac{22}{7}$.*)*

1. The radius of the base of a cylinder is 5 inches, and its curved surface is 440 square inches; find the volume.

2. The diameter of a cylinder is 1 ft. 2 in., and its volume is 1 c. ft. 1352 c. in., find its whole surface in square feet.

3. The weight of a cylindrical granite column is $8\frac{1}{4}$ tons, and its diameter is 2 feet. If a cubic foot of granite weighs 168 lbs., find the cost of polishing the curved surface of the column at the rate of 1*s.* 6*d.* a square foot.

4. Find the curved surface of a cylinder whose height is double of its diameter, and whose volume is 539 cubic inches.

5. How many revolutions are made by a cylindrical roller whose length is 6 feet and diameter $3\frac{1}{2}$ feet, in rolling a square field of $2\frac{1}{2}$ acres?

6. If the weights of equal volumes of iron and copper are in the ratio of 7 : 8, find the edge of a cube of copper equal in weight to a solid iron cylinder whose height is 4 inches and diameter 1 ft. 10 in.

(In the following Examples suppose 1 c. foot of water weighs 1000 oz.; 1 gallon = 277·25 c. inches; $\pi = 3{\cdot}1416$.*)*

7. The cylinder of a common pump is 6 inches in diameter; what must be the length of beat of the piston, if 8 beats are needed to raise 10 gallons?

8. A cubic inch of gold is drawn into a wire 1000 yards long. Find (to the nearest thousandth of an inch) the diameter of the wire.

9. Water flows through a cylindrical pipe 2·4 inches in internal diameter at the rate of 80 feet per minute. Find (to the nearest second) how long it would take to fill a vessel capable of holding 5 cwt. of water.

10. Three hundred and sixty gallons of water are drawn off from a cistern in half-an-hour by a cylindrical pipe. If the water flows at a uniform rate of 72 feet per minute, find (to the nearest hundredth of an inch) the internal diameter of the pipe.

11. A reservoir, standing on a base containing 7240 square inches, is supplied at a uniform rate by a cylindrical pipe 1·8 inches in diameter. At what rate (in feet per minute) must water flow through the pipe, if the depth in the reservoir is increased 8 inches in a quarter of an hour?

12. From a prism whose length is 10 inches, and whose transverse section is a regular hexagon on a side of 8 inches, the greatest possible cylinder is cut having the same axis as the prism. Find (in square and cubic inches correct to two places of decimals) the surface and volume of the cylinder.

13. A copper wire $\frac{1}{10}$ inch in diameter is evenly wound about a cylinder, whose length is 6 inches and diameter 9·9 inches, so as to cover the curved surface. Find the length and weight of the wire, if 1 cubic inch of copper weighs 5·1 oz.

(Hollow Cylinders.)

14. Find to the nearest ton what weight of stone will be required to line a semicylindrical tunnel 30 feet in internal diameter and 120 yards long. The lining is to be 15 inches thick, and 4 per cent. of the volume of the lining is to be deducted for cement. One cubic foot of the stone employed weighs 170 lbs.

15. Find (to the nearest tenth of a foot) what length of piping $1\frac{1}{2}$ inches in external diameter, and $\frac{1}{4}$ inch thick, could be made from a cubical block of lead whose edge is 1 foot. And find the weight of 21 feet of such piping, if a cubic foot of lead weighs 709·1 lbs.

16. The volume of metal in a cylindrical tube is 502·656 cubic inches. If the length is 8 inches, and its external diameter 1 foot, find its thickness.

17. A hollow cylindrical iron roller weighs 1314$\frac{3}{4}$ lbs.; if its length is 4 feet, and its external diameter 2 ft. 8 in.; find its thickness, having given 1 cubic foot of iron weighs 486 lbs.

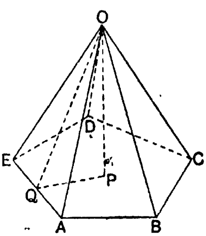
18. The whole surface of a cylindrical tube is 264 square inches: if its length is 5 inches, and its external radius 4 inches, find its thickness. $(\pi = \frac{22}{7}.)$

CHAPTER XVI.

THE PYRAMID.

97. Definition. A **pyramid** is a solid bounded by plane faces, of which one, called the *base*, is any rectilineal figure, and the rest are triangles having a common vertex at some point not in the plane of the base.

In the pyramid represented in the diagram, the figure ABCDE is the base; and the triangular faces OAB, OBC, *&c.*, have the common vertex O.



A pyramid is said to be *right*, when a perpendicular dropped from the vertex on the base meets the base at its central point (*i.e.* the centre of its inscribed or circumscribed circle, if the base is a regular figure, or the intersection of its diagonals, if the base is a rectangle).

Thus the pyramid shown in the diagram is right, because the perpendicular OP, drawn to the base from O, meets the base at P its central point. The perpendicular OP is the *height* of the pyramid; and OQ, which bisects the edge AE at right angles, is called the *slant height*. It is important to observe that OP is at right angles to any line, such as PQ, drawn in the plane of the base from P. [Def. 2, Art. 78.]

The sum of the triangular faces OAB, OBC, OCD, &c., is called the *slant surface* of the pyramid. In a right pyramid, standing on a *regular* base, the triangles OAB, OBC, OCD, &c., are all equal.

98. *To find the surface and volume of a right pyramid standing on a regular base of* n *sides.*

In the last figure let each side of the base $= a$ units of length; let the height OP $= h$, and the *slant* height OQ $= l$; and let the area of the base contain E square units. Then the perimeter of the base $= na$.

$\therefore$ Now the slant surface $= n$-times $\triangle OEA$

$$= n \times \frac{1}{2} AE \cdot OQ$$

$$= \frac{1}{2} na \cdot l \text{ square units;}$$

$\therefore$ (i) *slant surface of pyramid*

$$= \frac{1}{2} (\textit{perimeter of base}) \times (\textit{slant height}).$$

The *whole* surface = the slant surface + the area of the base.

Again, it may be shewn that the volume of a pyramid is ONE-THIRD the volume of the prism on the same base and of the same height,

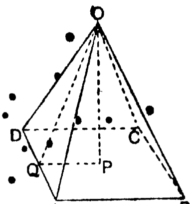
$\therefore$ (ii) *volume of pyramid* $= \frac{1}{3}$ (*area of base*) $\times$ *height*

$$= \frac{1}{3} Eh \text{ cubic units.}$$

The volume of an *oblique* pyramid is also given by the formula

$\frac{1}{3}$ (*area of base*) $\times$ (*perpendicular height*). Art. 91. Note.

Example. Find the slant surface and volume of a pyramid 21 inches high, standing on a square base whose side is 40 inches.



Here $h = OP = 21$ inches;

$a = AD = 40$ inches;

$\therefore PQ = \frac{1}{2} AB = 20$ inches.

From the $\triangle OPQ$, right-angled at P,

$OQ^2 = OP^2 + PQ^2 = (21)^2 + (20)^2 = 841.$

$\therefore OQ$ (or l) $= \sqrt{841} = 29$ inches.

Area of $\triangle ODA = \frac{1}{2} AD \cdot OQ$

$$= \frac{1}{2} \times 40 \times 29 \text{ sq. in.} = 580 \text{ sq. in.}$$

Hence (i) *the slant surface* $= 4 \triangle \text{ODA} = 4 \times 580$ sq. in.

$= 2320$ sq. in.

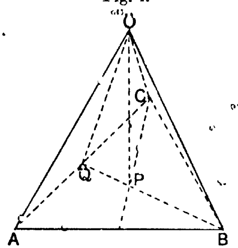
(ii) *The volume* $= \frac{1}{3}$ *(area of base)* $\times$ *height*

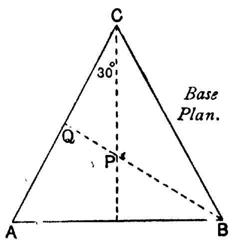
$= \frac{1}{3}(40)^2 \times 21$ c. in.

$= 11200$ c. in.

99. Definition. A **tetrahedron** is a pyramid on a triangular base: it is thus contained by *four* triangular faces.

In a *regular* tetrahedron the faces are equal and equilateral triangles.





To find the height, surface and volume of a regular tetrahedron whose edge is given.

Let OABC be a regular tetrahedron, each of whose edges is $2m$.

OP is the perpendicular from O, P being the central point of the equilateral triangle ABC.

Then, $\text{OQ} = \text{BQ} = m\sqrt{3}$; Art. 25.

and $\text{PQ} = \text{QC} \tan 30° = \frac{m}{\sqrt{3}}$. See fig. ii.

And from the $\triangle \text{OPQ}$, right-angled at P,

$$\text{OQ}^2 = \text{OP}^2 + \text{PQ}^2; \text{ or, } 3m^2 = \text{OP}^2 + \frac{m^2}{3}.$$

Hence (i) *height* $OP = 2m\sqrt{\frac{2}{3}}$;

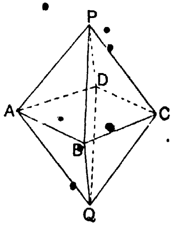
(ii) *whole surface* $= 4 \,.\, \triangle ABC = 4m^2\sqrt{3}$; Art. 25.

(iii) *volume* $= \frac{1}{3} \,.\, \triangle ABC \times OP$

$$= \frac{1}{3} \,.\, m^2\sqrt{3} \,.\, 2m\sqrt{\frac{2}{3}} = \frac{2}{3}m^3\sqrt{2}.$$

NOTE. A **regular octahedron** is a solid contained by *eight* equal equilateral triangles.

It will be seen by reference to the figure that a regular octahedron consists of two pyramids on opposite sides of a common square base ABCD. Hence its height and volume may be calculated by the method of the present article.



EXAMPLES. XVI. A.

ON PYRAMIDS.

[*Elementary Course.*]

(*Surfaces.*)

1. A right pyramid, 3 feet high, stands on a square base whose side is 8 feet. Find the area of one of the triangular faces.

2. Find the *whole* surface of a right pyramid of which the height is two feet, and the base a square on a side of 1 ft. 8 in.

3. Find the *slant* surface of a right pyramid one foot high, standing on a rectangular base whose length and breadth are 5 ft. 10 in. and 10 inches.

4. Find the cost of polishing the *slant* surface of a right pyramid, 6 ft. 5 in. high, standing on a square base, each side of which measures 6 feet, at the rate of 9*d.* per square foot.

(Volumes.)

5. Find the volumes of the pyramids in which

(i) the base is a square on a side of 5 inches, and the height is 6 inches,

(ii) the base is a rectangle measuring 8 inches by 4 inches, and the height is 1 foot,

(iii) the base is a triangle whose sides are 13 inches, 12 inches, 5 inches; and the height is 9 inches.

6. Find the heights of the pyramids in which

(i) the volume is 84 cubic inches, and the base a square on a side of 6 inches.

(ii) the volume is 1224 cubic inches, and the base a triangle whose sides are 4 ft. 3 in., 3 ft. 1 in., and 1 ft. 8 in.

7. A pyramid, standing on a square base, is 15 inches in height. If the volume is 320 cubic inches, find the side of the base.

8. The height of a pyramid standing on a rectangular base is 2 ft. 8 in., and its volume is 5 cubic feet. If the length of the base is 3 ft. 9 in., find its breadth.

(Miscellaneous.)

9. Find the weight of a granite pyramid 9 feet high, standing on a square base whose side is 3 ft. 4 in. [Given 1 cubic foot of granite weighs 165 lbs.]

10. A tower, whose ground plan is a square on a side of 30 feet, is furnished with a pyramidal roof 8 feet high. Find the cost of covering the roof with sheet lead at the rate of 8*d.* per square foot.

11. A marble column, 6 feet high, having a square cross-section on a side of 15 inches, is surmounted by a pyramid whose height is 1 ft. 6 in. Find the weight of the whole, having given 1 cubic foot of marble weighs 170 lbs.

12. The base of a pyramid is square, and covers an area of 1764 square feet, and its volume is 11760 cubic feet; find its slant surface.

EXAMPLES. XVI. B.

On Pyramids.

[*Higher Course.*]

(*Edges and Surfaces.*)

1. Find the slant surface of a right pyramid having the same base and height as a cube whose edge is 10 inches.

2. A right pyramid stands on a rectangular base whose sides are 3 ft. 6 in. and 2 feet; if each of the slant edges measures 2 ft. 5 in., find the height of the pyramid.

3. The base of a right pyramid is an equilateral triangle on a side of 1 foot, and its height is 7 inches: find the area of one of its slant faces.

4. A right pyramid stands on a regular hexagonal base whose side is 8 inches. If the height is 4 inches, find the area of one of its triangular faces.

5. The slant faces of a right pyramid, standing on a square base, are equilateral triangles, each side of which is 16 inches. Find the height and slant surface of the pyramid.

6. A right pyramid stands on a hexagonal base, each side of which is 14 inches. If each slant *edge* is 21 inches, find the height and slant surface of the pyramid.

(*Volumes.*)

Take $\sqrt{3} = 1{\cdot}732$.

7. From a rectangular block of deal, standing on a square base whose side is 1 ft. 4 in., a pyramid is cut having the same base and height; if the wood that is cut away weighs 152 lbs., find the height of the pyramid. [Given 1 cubic foot of deal weighs 912 ounces.]

8. Find the volume of a pyramid whose base is an equilateral triangle on a side of 1 ft. 6 in., the height being equal to the perpendicular drawn from a vertex of the base to the opposite side of the base.

9. Find the height of a pyramid, of which the volume is 623·52 cubic inches, and the base a regular hexagon on a side of 1 foot.

10. A right pyramid stands on a square base containing 300 square inches; and the perpendicular height of the pyramid is half its slant height. Find its volume.

11. The base of a right pyramid is a regular hexagon on a side of 20 inches, and its slant faces are inclined to the base at an angle of 60°. Find the volume.

12. The base of a pyramid, 6 inches high, is an equilateral triangle; if the volume is 346·41 cubic inches, find the length of each side of the base.

(*Miscellaneous.*)

13. From a solid cylinder the greatest possible pyramid on a square base and of the same height is cut: express its volume as the decimal (to four figures) of the volume of the given cylinder.

14. OA, OB, OC, are conterminous edges of a cube, whose edge is a inches. Find the volume of the pyramid OABC, the area of the triangle ABC, and the length of the perpendicular drawn from O on the plane of ABC.

15. Three conterminous edges of a rectangular solid OA, OB, OC, measure respectively a, b and c inches: find the volume of the pyramid OABC, the area of the triangle ABC, and the length of the perpendicular drawn from O to the plane of the triangle ABC.

16. The corners of a cube are cut off by planes which pass through the middle points of each set of conterminous edges. If the edge of the cube was originally 2 feet, find the volume and surface of the resulting figure.

17. Find the edge of the greatest cube that can be cut from a right pyramid h inches high, standing on a square base whose side is a inches, one face of the cube being in the plane of the base of the pyramid.

18. From a right pyramid, whose height is 9 inches, and whose base is a square on a side of 6 inches, a rectangular solid is to be cut, so that its base lies in the base of the pyramid, and its upper vertices in the slant edges of the pyramid. The base of the rectangular solid is to be a square on a side of 2 inches; find its height.

19. Find (to the nearest tenth of an inch) the edge of a regular tetrahedron whose volume is 117·851 cubic inches.

20. Find the surface of a regular tetrahedron, if the perpendicular from one vertex to the opposite face is 5 inches.

21. Find the volume of a regular octahedron each edge of which is a feet.

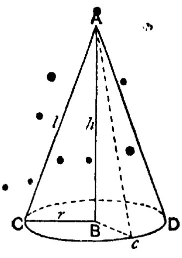
22. The volume of a regular octahedron is 471·41 cubic feet; find the length of each edge.

CHAPTER XVII.

THE RIGHT CIRCULAR CONE.

100. Definition. A **right circular cone** is a solid described by the revolution of a right-angled triangle about one of its sides (containing the right angle) which remains fixed.

Thus if the right-angled triangle ABC revolves about the side AB, it describes the cone represented in the figure. AB is said to be the *axis*, or *height*, of the cone; and the angle CAD is called its *vertical angle*.



The whole surface of the cone consists of the *curved surface* described by AC, and of the *circular end*, or *base*, described by BC.

A similarity in character between the cylinder and prism has been pointed out on page 135. The beginner will now observe that the same relation exists between the cone and pyramid; a pyramid becoming a cone, when its base is a circle instead of a rectilineal figure. Accordingly the rules for finding the surface and volume of a cone correspond with those already given for the pyramid.

NOTE. The above definition refers only to a *right circular cone*. Cones may exist whose axes are not perpendicular to their bases, and whose bases are not circular. Such cones, however, are beyond the scope of the present text-book.

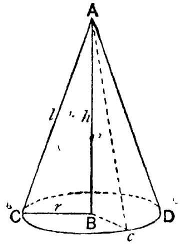
101. *To find the surface and volume of a cone.*

In the adjoining figure let the height AB $= h$ units of length, the radius of the base BC $= r$, and the *slant height* AC $= l$.

Then from the right-angled $\triangle$ ABC,

$$AC^2 = AB^2 + BC^2;$$

or,
$$l^2 = h^2 + r^2; \quad l = \sqrt{h^2 + r^2}.$$



(i) *The curved surface of cone*

$$= \frac{1}{2}(\textit{circumference of base}) \times (\textit{slant height})$$

$$= \frac{1}{2} \cdot 2\pi r \times l$$

$$= \pi r l \text{ square units.}$$

The *whole surface* of cone = curved surface + area of base

$$= \pi r l + \pi r^2$$

$$= \pi r (l + r).$$

Again, it may be shewn that the volume of a cone is *one-third* the volume of the cylinder on the same base and of the same height;

$\therefore$ (ii) *the volume of cone* $= \frac{1}{3}$ *(area of base)* $\times$ *height*

$$= \frac{1}{3}\pi r^2 h \text{ cubic units.}$$

NOTE. The formula for the curved surface of a cone may be illustrated by bending round a piece of paper ACD, cut in the form of the sector of a circle, until the lines AC and AD coincide. In this way the sector may be made to take the shape of the curved surface of a cone; CD, the arc of the sector, becoming the *circumference* of the base, and AC, the radius of the sector, becoming the *slant height* of the cone.

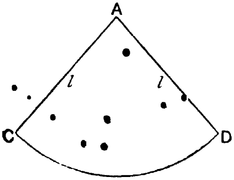


Fig. i.

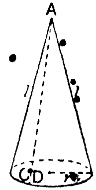


Fig. ii.

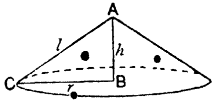
Thus the curved surface of the cone = area of sector

$$= \frac{1}{2} \text{arc} \times (\text{radius of sector}) \qquad \text{Art. } 53.$$

$$= \frac{1}{2} (\text{circumference of base}) \times (\text{slant height})$$

$$= \frac{1}{2} \cdot 2\pi r \times l = \pi r l.$$

Example i. Find the *slant* height, the whole surface, and the volume of the cone whose height is 1 foot, and the diameter of whose base is 5 ft. 10 in. $\left(\pi = \frac{22}{7}.\right)$



Here $h = 12$ inches; $r = 35$ inches;

and $l^2 = h^2 + r^2 = 144 + 1225 = 1369$.

(i) $\therefore \; l = \sqrt{1369} = 37$ inches.

(ii) *Whole surface = curved surface + area of base*

$$= \pi r l + \pi r^2 = \pi r (l + r)$$

$$= \frac{22}{7} \times 35 \times 72 \text{ sq. in.} = 55 \text{ sq. ft.}$$

(iii) *Volume* $= \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} \cdot \frac{22}{7} \times (35)^2 \times 12 \text{ c. in.} = 8{\cdot}91\ldots \text{ c. ft.}$$

**Example* ii. Water flows at the rate of 20 feet per minute from a cylindrical pipe ·25 inches in diameter. How long would it take to fill a conical vessel whose diameter at the surface is 10 inches, and depth 9 inches? ($\pi = 3{\cdot}1416$.)

Suppose it takes x minutes to fill the vessel

In *one* minute a cylindrical column of water $\frac{1}{8}$ inch in radius and 240 inches long enters the vessel;

$\therefore$ in x minutes the volume of water passed

$$= x \times \pi \times \left(\frac{1}{8}\right)^2 \times 240 \text{ c. in.}$$

$$= x \,.\, \pi \times \frac{15}{4} \text{ c. in.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

And the volume of the cone

$$= \frac{1}{3} \pi \times 5^2 \times 9 \text{ c. in.} = 75\pi \text{ c. in.} \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

$$\therefore x \,.\, \pi \times \frac{15}{4} = 75\pi;$$

$$\therefore x = 20 \text{ minutes.}$$

[Obs. A point worth notice in this example is that the value of π is not to be substituted at the stages marked (i) and (ii), as at the final stage it is cancelled from both sides of the equation.]

**Example* iii. The height of a cone is $3\frac{1}{2}$ inches, and its curved surface is four times the area of the base. Find its volume to the nearest hundredth of a cubic inch. $\left(\pi = \frac{22}{7}.\right)$

Here $h = \frac{7}{2}$; required to find r.

By hypothesis $\pi r l = 4\pi r^2$,

$$\therefore l = 4r,$$

$$\therefore l^2 = 16r^2; \text{ or, } h^2 + r^2 = 16r^2.$$

And since $h = \frac{7}{2}$, $\therefore r^2 = \frac{49}{60}$.

Now volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times \frac{49}{60} \times \frac{7}{2}$ c. in.

$$= 2{\cdot}99\ldots \text{ c. in.} = 3 \text{ c. in. nearly.}$$

EXAMPLES. XVII. A.

ON CONES.

[*Elementary Course.*]

In the following Examples take $\pi = \frac{22}{7}$.

(*Surfaces.*)

1. Find the curved surfaces of the cones of which

(i) the slant height is 1 ft. 9 in., and the radius of the base 5 inches;

(ii) the vertical height is 1 foot, and the radius of the base 2 ft. 11 in.;

(iii) the vertical height is 5 feet, and the diameter of the base 15 ft. 2 in.

(iv) the slant height is 2 ft. 1 in., and the vertical height 2 feet.

2. The curved surface of a cone is 396 square inches, and the radius of the base is 6 inches; find its slant height.

3. The curved surface of a cone is 2 sq. ft. 42 sq. in., and its slant height is 1 ft. 3 in.; find the radius of the base.

4. Find the cost of canvas for a conical tent, whose height is 15 feet, and the diameter of whose base is 16 feet, at 1*s.* 3¾*d.* per square yard.

5. How many square yards of canvas will be required for a conical tent 24 feet high, if the area of the base is 154 square feet?

(*Volumes.*)

6. Find the volumes of the cones of which

(i) the radius of the base is 3 inches, and the height 7 inches;

(ii) the diameter of the base is 1 ft. 2 in., and the height 1 foot;

(iii) the height is 2 feet, and the *slant* height 2 ft. 1 in.;

(iv) the diameter of the base is 6 feet, and the *slant* height 7 ft. 1 in.

7. Find the height of the cones of which

(i) the volume is 88 cubic inches, and the radius of the base is 2 inches;

(ii) the volume is 1 c. ft. 648 c. in., and the diameter of the base is 1 ft. 3 in.

8. Find the radius of the base and the slant height of the cones of which

(i) the volume is 3 c. ft. 96 c. in., and the height 2 ft. 11 in.;

(ii) the volume is 36 c. ft. 1680 c. in., and the height 4 ft. 8 in.

9. A conical vessel measuring 2 ft. 4 in. across the top, and $4\frac{4}{5}$ feet deep, is placed with its axis vertical and vertex downwards. How many gallons will it hold, given 1 cubic foot $= 6\frac{1}{4}$ gallons?

10. Find (to the nearest pound) the weight of an iron cone, the height being 1 ft. 2 in., and the diameter of the base 1 foot; given that one cubic foot of iron weighs $486\frac{3}{4}$ lbs.

(Miscellaneous.)

11. A cylindrical granite pillar 20 feet high and 16 inches in diameter, is surmounted by a cone 3 feet in height; find the weight of the whole, having given one cubic foot of granite weighs 165 lbs.

12. The weight of a brass cone is 55 cwt., and the diameter of its base is 3 ft. 6 in.; if one cubic foot of brass weighs 500 lbs., find the height of the cone in feet.

13. Find (to the nearest hundredth of an inch) the radius of the base of a cone whose height and volume are the same as those of a cylinder 1 ft. 8 in. in diameter.

14. The base of a cone is 4 sq. ft. 40 sq. in., and its height is 4 feet; find its volume, and the area of its curved surface.

15. From a cylinder whose height is 3·3 inches, and diameter 11·2 inches, a conical cavity of the same height and base is hollowed out. Find the whole surface of the remaining solid.

EXAMPLES. XVII. B

ON CONES.

[*Higher Course.*]

In the following Examples take $\pi = 3{\cdot}1416$.

(*Surfaces.*)

1. Into each end of a solid cylinder, whose length is 10 inches and diameter 8 inches, a conical cavity is bored; if the diameter of each cavity is 6 inches, and its depth is 4 inches, find (to the nearest hundredth of a square inch) the whole surface of the remaining solid.

2. The area of the base of a cone is 78·54 square inches, and its height is one foot; find its curved surface, to the nearest hundredth of a square inch.

3. Find (to the nearest hundredth of a square inch) the *whole* surface of the greatest cone that can be cut from a solid cube whose edge is 1 ft. 8 in., the base of the cone being in the same plane as the base of the cube.

4. The height of a cone is 27 inches, and its curved surface is seven times the area of its base. Find (to the nearest hundredth of an inch) the radius of its base.

(*Volumes.*)

In the following Examples take 1 *gallon* = 277·25 *cubic inches.*

5. Find the height of a cone, the radius of whose base is 1 ft. 9 in., and whose volume is equal to that of a cylinder of diameter 3 feet, and height 12 ft. 3 in.

6. Find the radius of the base of a cone, whose height is 9 feet, and whose volume is equal to that of a cylinder 9 inches high, and 5 feet in diameter.

7. Find approximately in gallons, the capacity of a conical vessel having a diameter of 3 ft. 4 in. at the surface, and a depth of 2 feet.

8. A conical vessel has a diameter at the surface of 30 inches, and a depth of one foot. Find how long it would take to fill it with water flowing at the rate of 50 feet per minute from a cylindrical pipe ·6 inches in diameter.

9. A conical vessel has a diameter at the surface of 18 inches; find its depth to the nearest hundredth of an inch, if its capacity is 10 gallons.

(Miscellaneous.)

In the following Examples take $\pi = \frac{22}{7}$.

10. The curved surface of a cylinder is 2640 square inches, and its volume is 26400 cubic inches; find the curved surface of the right cone which has the same base and height as the cylinder.

11. Find the volume of a cone whose curved surface is 550 square inches, and the radius of whose base is 7 inches.

12. The height of a cone is 7 inches, and its curved surface is 3 times the area of its base. Find its volume to the nearest hundredth of a cubic inch.

13. Find the curved surface of a cone whose height is 20 inches, and whose volume is 9240 cubic inches.

14. A sector of a circle of radius 2 feet and vertical angle 60°, is wrapped into the form of a right cone. Find (to the nearest hundredth of an inch) the height of the cone.

15. A right-angled triangle, whose remaining angles are 60° and 30°, revolves about its hypotenuse, which is 6 inches in length. Find (to the nearest hundredth of an inch) the volume of the solid figure thus described.

16. Find the dimensions of a right cone whose vertical angle is 120°, and whose volume is 4 c. ft. 1712 c. in.

17. A cone, whose vertical angle is 60°, is pressed (vertex downwards) into a vessel full of water. When 6 gallons have overflowed, find (to the nearest tenth of an inch) the depth of the vertex below the surface of the water.

18. A vertical cylindrical vessel of radius 1 foot, is partly filled with water, and into it is plunged a solid cone, whose height is equal to the diameter of its base. If, when the cone is completely immersed, the water rises 4 inches, find the dimensions of the cone (to the nearest tenth of an inch).

CHAPTER XVIII.

THE FRUSTUM OF A PYRAMID AND CONE.

102. Definition. A frustum (or slice) of a pyramid or cone is the part contained between the base and a plane drawn parallel to the base.

Fig. i.

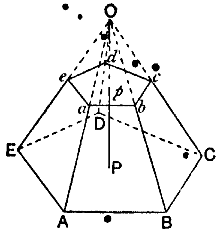
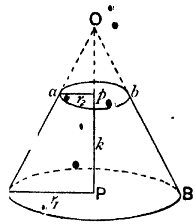


Fig. ii.



Thus the frustum of the pyramid represented in fig. i. is that part which is included between the base ABCDE and the parallel plane *abcde*.

The frustum of the cone (fig. ii.) is contained between the base AB and the parallel plane *ab*.

The figures ABCDE, *abcde* in fig. i. and the figures AB, *ab* in fig. ii. are called the *ends* of the frustum. The ends of the frustum of a pyramid are *similar figures*. The ends of the frustum of a cone are *circles*.

The slant surface of the frustum of a pyramid is made up of trapeziums. If the pyramid is *right*, and the base ABCDE is a *regular* polygon, these trapeziums are all equal.

NOTE. *The areas of the ends of a frustum are proportional to the squares of the perpendiculars* OP, O*p*. (Euclid, p. 424.)

103. *To find the slant surface and volume of the frustum of a right pyramid on a regular base of* n *sides.*

In the above figure, let the ends ABCDE, *abcde* contain E_1 and E_2 square units respectively; and let the thickness P$p = k$ units of length. Let a_1 be the length of each side

of the end ABCDE, and a_2 of the end *abcde*. Also let l be the distance between any two corresponding sides of the ends, such as AE and *ae*; this may be called the *slant thickness* of the frustum.

Then the slant surface $= n$ times trapezium AaeE

$$= n \times \frac{1}{2}(a_1 + a_2)\, l \qquad \text{Art. 34}$$

$$= \frac{1}{2}(na_1 + na_2)\, l \text{ sq. units,}$$

$\therefore$ (i) *Slant surface of frustum* $= \frac{1}{2}$(*sum of perimeters of ends*) $\times$ (*slant thickness*).

(ii) *Volume of frustum* $= \frac{1}{3} k \left[E_1 + \sqrt{E_1 E_2} + E_2 \right]$ cubic units.

The formula (ii) gives the volume of the frustum of *any* pyramid, regular or otherwise.

104. *To find the curved surface and volume of the frustum of a right cone.*

In fig. ii. of page 159, let r_1 and r_2 be the radii of the two ends AB and *ab* respectively, and let the thickness $Pp = k$. Let the *slant thickness* $Aa = l$. Then if E_1 and E_2 denote the areas of the ends, we have $E_1 = \pi r_1^2$, and $E_2 = \pi r_2^2$.

$\therefore$ (i) *Curved surface of frustum of cone*

$$= \frac{1}{2}(\textit{sum of circumferences of ends}) \times (\textit{slant thickness})$$

$$= \frac{1}{2}(2\pi r_1 + 2\pi r_2)\, l = \pi (r_1 + r_2)\, l \text{ sq. units.}$$

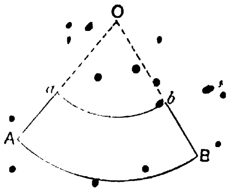
(ii) *Volume of frustum* $= \frac{1}{3} k \left[E_1 + \sqrt{E_1 E_2} + E_2 \right]$

$$= \frac{1}{3} k \left[\pi r_1^2 + \sqrt{\pi r_1^2 \cdot \pi r_2^2} + \pi r_2^2 \right]$$

$$= \frac{\pi}{3} k \left[r_1^2 + r_1 r_2 + r_2^2 \right] \text{ cubic units.}$$

*NOTE 1. The formula giving the curved surface of the frustum of a cone may be thus established.

Let OAB and Oab be two sectors of circles having a common vertical angle. Then by bending round the figure ABba until Aa and Bb coincide, it may be made to take the form of the curved surface of a frustum of a cone, the arcs AB and ab becoming the circumferences of the ends of the frustum.



Let $\text{OA}=l_1$, $\text{O}a=l_2$, and $\text{A}a=l$; so that $l_1-l_2=l$, the slant thickness of the frustum.

Then $$\frac{\text{arc AB}}{l_1}=\frac{\text{arc } ab}{l_2}=\theta.$$ Art. 50.

$$\therefore \text{ arc AB}=l_1\theta, \text{ and arc } ab=l_2\theta.$$

Now *curved surface of frustum* = sector OAB − sector Oab

$$=\frac{1}{2}\text{arc AB}\times l_1-\frac{1}{2}\text{arc } ab\times l_2 \qquad \text{Art. 53}$$

$$=\frac{1}{2}(l_1^2-l_2^2)\,\theta=\frac{1}{2}(l_1-l_2)(l_1+l_2)\,\theta$$

$$=\frac{1}{2}l\,(l_1\theta+l_2\theta)=\frac{1}{2}l\,(\text{arc AB}+\text{arc } ab)$$

$$=\frac{1}{2}l\times(\textit{sum of circumferences of ends}).$$

*NOTE 2. The expression $\frac{1}{3}k\,[\text{E}_1+\sqrt{\text{E}_1\text{E}_2}+\text{E}_2]$ for the volume a frustum of a pyramid or cone may be similarly obtained.

Let $\text{OP}=h_1$, $\text{O}p=h_2$; so that $h_1-h_2=k$. See fig. p. 159.

From Note p. 159, we have $\dfrac{\text{E}_1}{h_1^2}=\dfrac{\text{E}_2}{h_2^2}=\mu$ (say),

$$\therefore \text{E}_1=\mu h_1^2, \text{ and } \text{E}_2=\mu h_2^2.$$

And volume of frustum $=\frac{1}{3}\text{E}_1h_1-\frac{1}{3}\text{E}_2h_2=\frac{1}{3}(h_1^3-h_2^3)\,\mu$

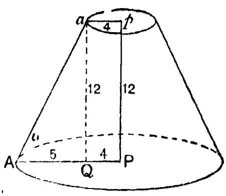
$$=\frac{1}{3}(h_1-h_2)\{h_1^2+h_1h_2+h_2^2\}\mu$$

$$=\frac{1}{3}k\,\{\mu h_1^2+\sqrt{\mu h_1^2\cdot\mu h_2^2}+\mu h_2^2\}$$

$$=\frac{1}{3}k\,[\text{E}_1+\sqrt{\text{E}_1\text{E}_2}+\text{E}_2].$$

Example i. Find the curved surface and volume of the frustum of a cone whose thickness is one foot, and whose ends are 1 ft. 6 in. and 8 inches in diameter.

Here $k = 12$ inches, $r_1 = 9$ inches, $r_2 = 4$ inches.



In the annexed figure draw aQ perp. to AP. Then $aQ = 12$ inches, and $AQ = 5$ inches; required Aa, or l.

$$Aa^2 = aQ^2 + AQ^2;$$

or $$l^2 = 12^2 + 5^2 = 169.$$

$$\therefore\ l = 13 \text{ inches.}$$

(i) Curved surface $= \pi (r_1 + r_2)\, l$

$$= \frac{22}{7} \times 13 \times 13 \text{ sq. in.} = 531\tfrac{1}{7} \text{ sq. in.}$$

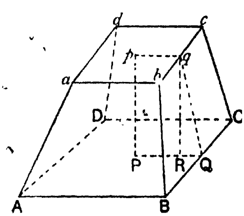
(ii) Volume $$= \frac{\pi}{3} k\, (r_1^2 + r_1 r_2 + r_2^2)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 12 \times (81 + 36 + 16) \text{ cubic inches}$$

$$= 1672 \text{ cubic inches.}$$

Example ii. Find the slant surface and volume of the frustum of a pyramid whose thickness is 1 ft. 3 in., and whose ends are squares on sides of 3 ft. 4 in. and 2 ft. respectively.

Referring to the figure we see that



$$PQ = \frac{1}{2} AB = 20 \text{ inches,}$$

$$pq = \frac{1}{2} ab = 12 \text{ inches,}$$

$$Rq = Pp = k = 15 \text{ inches,}$$

$$RQ = PQ - PR = 8 \text{ inches}$$

Required the length of Qq, or l.

$$Qq^2 = Rq^2 + RQ^2 = 15^2 + 8^2 = 289;$$

$$\therefore\ l = 17 \text{ inches.}$$

(i) Slant surface $= 4$ times trapezium $\mathrm{BC}cb$

$$= 4 \times \frac{1}{2} (\mathrm{BC} + bc)\, \mathrm{Q}q \qquad \text{Art. 34}$$

$$= 4 \times \frac{1}{2} \times (40 + 24) \times 17 \text{ sq. inches}$$

$$= 2176 \text{ sq. inches} = 15 \text{ sq. ft. } 16 \text{ sq. in.}$$

(ii) $\mathrm{E}_1 =$ sq. on $\mathrm{AB} = 1600$ sq. inches,

$\mathrm{E}_2 =$ sq. on $ab = 576$ sq. inches

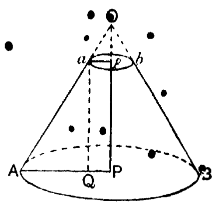
$$\text{Volume} = \frac{k}{3} [\mathrm{E}_1 + \sqrt{\mathrm{E}_1\mathrm{E}_2} + \mathrm{E}_2]$$

$$= \frac{15}{3} [1600 + \sqrt{1600 \times 576} + 576] \text{ cubic inches}$$

$$= 5 [1600 + 40 \times 24 + 576] \text{ cubic inches}$$

$$= 15680 \text{ cubic inches} = 9 \text{ c. ft. } 128 \text{ c. in.}$$

Example iii. A cone 12 inches in height, and 16 inches in diameter at the base, is cut by a plane parallel to the base and 9 inches distant from it. Find the curved surface and the volume of the frustum so formed.



Here r_1 and k are given, also the height of the complete cone.

We have first to find the value of r_2.

From the similar triangles APO, apO,

$$\frac{r_1}{\mathrm{OP}} = \frac{r_2}{\mathrm{O}p}; \text{ or, } \frac{8}{12} = \frac{r_2}{3}.$$

$$\therefore r_2 = 2;$$

hence $\mathrm{AQ} = 6$.

Aa, or l, is found from the right-angled triangle AQa;

for $l = \sqrt{\mathrm{AQ}^2 + a\mathrm{Q}^2} = \sqrt{117} = 10{\cdot}81$ nearly,

(i) Curved surface $= \pi (r_1 + r_2)\, l$

$$= \frac{22}{7} \times 10 \times 10{\cdot}81 \text{ sq. inches} = 339{\cdot}74 \text{ sq. in.}$$

(ii) Volume $= \frac{\pi}{3} k \{r_1^2 + r_1 r_2 + r_2^2\}$

$$= \frac{22}{21} \times 9 \times \{64 + 16 + 4\} \text{ c. inches} = 792 \text{ c. inches.}$$

EXAMPLES. XVIII. A.

ON THE FRUSTUM OF A PYRAMID AND CONE.

[*Elementary Course.*]

In the following Examples take $\pi = \frac{22}{7}$.

(*Surfaces.*)

1. Find the curved surface of the frustum of a cone whose *slant* thickness is 5 inches, and whose circular ends are 8 inches and 6 inches in diameter.

2. Find the curved surface of the frustum of a cone whose *slant* thickness is 1 ft. 3 in., and whose ends are 1 ft. 10 in. and 1 ft. 8 in. in diameter.

3. Find the curved surface of the frustum of a cone whose thickness is 1 ft. 3 in., and whose ends are 3 feet and 1 ft. 8 in. in diameter.

4. The ends of the frustum of a pyramid are squares on sides of 20 inches and 4 inches; if the frustum is 15 inches thick, find its slant surface.

5. An open vessel, in the shape of a frustum of a cone, has the following measurements: the upper and lower diameters are 2 ft. 4 in. and 1 ft. 2 in. respectively, and the depth is 2 feet; find the whole surface.

6. An open vessel, in the form of a frustum of a cone, is to be lined with thin metal which costs 3*s*. 6*d*. per square foot. Find the whole cost, if the upper and lower diameters of the vessel are 1 ft. 8 in. and 8 in., and its depth is 8 inches.

(*Volumes.*)

7. Find the volume of the frustum of a pyramid, the ends being squares on sides of 8 inches and 6 inches, and the thickness being 3 inches.

8. The ends of a frustum of a pyramid are rectangles, the base measuring 9 inches by 6 inches, and the top 3 inches by 2 inches; if the thickness is 5 inches, find the volume.

9. Find the volume of the frustum of a cone

(i) if the radii of the ends are 5 inches and 3 inches, and the thickness 1 foot,

(ii) if the ends are 2 ft. 6 in. and 1 ft. 6 in. in diameter, and the thickness 9 inches.

10. The ends of the frustum of a pyramid are triangles, the sides of the base measuring 13 inches, 12 inches, 5 inches, and the sides of the top $6\frac{1}{2}$ inches, 6 inches, $2\frac{1}{2}$ inches; if the thickness of the frustum is 8 inches, find the volume.

11. The *slant* thickness of a frustum of a cone is 5 inches, and the radii of its ends are 4 inches and 1 inch respectively: find its volume.

12. The *slant* thickness of a frustum of a cone is 2 ft. 1 in., and the diameters of its ends are 2 ft. 4 in. and 1 ft. 2 in. respectively; find its volume.

EXAMPLES. XVIII. B.

On the Frustum of a Pyramid and Cone.

[*Higher Course.*]

(*Surfaces.*)

1. The ends of a frustum of a pyramid are squares on sides of 20 inches and 4 inches: if the frustum is 15 inches thick, find its slant surface.

2. The ends of a frustum of a pyramid are squares on sides of 8 inches and 1·4 inch: if the thickness of the frustum is 5·6 inches, find its slant surface.

3. A cone, 31·5 inches in height, standing on a base 60 inches in diameter, is cut through by a plane parallel to the base and 21 inches distant from it: find (to the nearest hundredth of a square inch) the curved surface of the frustum so found. $[\pi = 3{\cdot}1416.]$

4. A conical vessel $7\frac{1}{2}$ inches deep and 20 inches across the top is completely filled with water. If sufficient water is now drawn off to lower its level by 6 inches, find (to the nearest hundredth of a square inch) the surface of the vessel thus exposed. $[\pi = 3{\cdot}1416.]$

5. Water is poured into a conical vessel, 12·6 inches deep and 24 inches across the top, until the surface of the water stands 10·5 inches above the vertex of the cone: find (to the nearest hundredth of a square inch) the surface of the vessel left uncovered.

6. The ends of a frustum of a cone are respectively 8 inches and 2 inches in diameter: if its curved surface is equal to the area of a circle whose radius is 5 inches, find the thickness of the frustum.

(*Volumes.*)

7. The ends of a frustum of a tetrahedron are equilateral triangles on sides of 10 inches and 8 inches: if the thickness of the frustum is 3 inches, find its volume to the nearest hundredth of a cubic inch.

8. The ends of a frustum of a pyramid are regular hexagons on sides of 6 inches and 4 inches: if the thickness of the frustum is 5 inches, find its volume to the nearest hundredth of a cubic inch.

9. A dyeing vat is in the form of a frustum of a square pyramid. Each side of the base measures 6 feet, and the slant faces are inclined to the base at an angle of 135°. If the vat is 1 yard deep, how many gallons will it hold, reckoning 1 cubic foot = $6\frac{1}{4}$ gallons?

10. The diameters of the ends of a frustum of a cone are 6 inches and 2 inches respectively, and its *slant* thickness is 2·9 inches: find its volume. [$\pi = \frac{22}{7}$.]

11. A cone, 20 inches in height, standing on a base of diameter 10 inches, is cut by a plane parallel to the base, and 6 inches distant from it. Find the volume of the frustum so formed. [$\pi = \frac{22}{7}$.]

12. A frustum 7 inches thick is cut by a plane from a cone whose height is 3 ft. 6 in. and base 1 foot in diameter. Find the volume of the frustum. [$\pi = \frac{22}{7}$.]

13. A cone 2 feet in height, standing on a base 1 ft. 8 in. in diameter, is cut by a plane parallel to the base so as to leave a frustum whose *slant* thickness is 1 ft. 1 in.: find the volume of the frustum. [$\pi = \frac{22}{7}$.]

(Miscellaneous.)

Suppose 1 *gallon* = 277·25 *cubic inches.*

14. A vessel in the form of a frustum of a cone contains 37 gallons. If the diameters of the two ends are 1 ft. 4 in. and 1 foot, find the depth. [$\pi = 3{\cdot}1416$.]

15. The ends of the frustum of a pyramid are regular hexagons on sides of 8 inches and 6 inches: if the thickness is 3 inches, find the slant surface to the nearest hundredth of a square inch.

16. A cone, whose height is n inches, is cut through by a plane parallel to the base and one inch distant from it: express the volume of the frustum so formed as a fraction of the volume of the whole cone.

17. A cone of height h is cut through by a plane parallel to its base: at what distance from the vertex must the cutting plane be, if the volume of the frustum thus formed is half that of the original cone?

18. Water is poured into a conical vessel, whose angle is 90°, until the surface stands 10 inches above the vertex. How many gallons must now be added to raise the surface by 2 inches?

[$\pi = \frac{22}{7}$.]

19. Into a conical vessel, whose angle is 60° and depth 9 inches, water is poured until the surface is 6 inches above the vertex. Find (to the nearest tenth of a pint) how much water must now be added so as to fill the vessel. [$\pi = \frac{22}{7}$.]

20. Water is poured into a conical vessel, whose angle is 60°, until the surface of the water is 6 inches above the vertex. A solid cube of metal is now dropped in, and is totally submerged: if the surface of the water rises $\frac{1}{10}$ inch, find the edge of the cube to the nearest tenth of an inch. [$\pi = \frac{22}{7}$.]

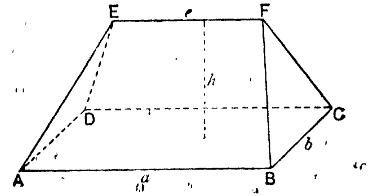
21. The volume of a frustum of a cone is 407 cubic inches, and its thickness is $10\frac{1}{2}$ inches. If the diameter of one end is 8 inches, find the diameter of the other end. [$\pi = \frac{22}{7}$.]

22. The frustum of a square pyramid is 6 inches thick, and the area of one end is four times the area of the other. If the volume is 350 cubic inches, find the dimensions of the ends.

CHAPTER XIX

THE WEDGE, FRUSTUM OF WEDGE, AND PRISMOID.

105. Definition. A **wedge** is a solid contained by five plane faces; the base is a rectangle, the two ends are triangles, and the two remaining faces are trapeziums having a common side, called the *edge*, which is parallel to the base.



In the diagram, ABCD is the rectangular base; and EF is the edge.

NOTE. If the edge EF = AB the length of the base, then the faces ABFE, DCFE are parallelograms, and the wedge is a triangular prism.

106. *To find the volume of a wedge.*

Let the length of the base = a units, the breadth = b, the height = h, and the edge EF = e.

Then *volume of wedge* $= \frac{hb}{6}(2a + e)$ cubic units.

The surface of a wedge is found by calculating separately the area of each of the faces. To do this, the *slant* heights of the faces (or the means of finding them) must be given.

NOTE. The formula for the volume of a wedge may be proved by dividing the solid into a prism and two pyramids by planes drawn through E and F perpendicular to the edge.

Example. Find the weight of a steel wedge whose base measures 8 inches by 5 inches; the height of the wedge being 6 inches. [Given 1 cubic inch of steel weighs 4·53 oz.]

Here $a = 8$ inches, $b = 5$ inches, $e = 4$ inches, $h = 6$ inches.

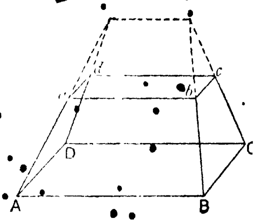
$$\textit{Volume of wedge} = \frac{hb}{6}(2a+e) = \frac{6 \times 5}{6}\{16+4\} \text{ c. in.} = 100 \text{ c. in.}$$

$$\therefore \text{ the } \textit{weight} = 100 \times 4{\cdot}53 \text{ oz.} = 453 \text{ oz.}$$

107. Definitions. 1. The **frustum of a wedge** is the part included between the base and a plane parallel to the base.

In the figure, ABCD is the rectangular base, and *abcd* represents the cutting plane. The end *abcd* is rectangular, and the remaining faces are trapeziums.

NOTE. The frustum of a wedge differs from the frustum of a pyramid in that the ends ABCD, *abcd* are *not similar*: hence the edges A*a*, B*b*, C*c*, D*d*, do not, if produced, meet at a point.

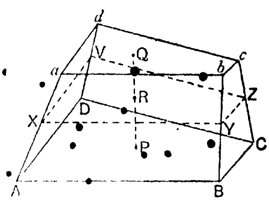


2. A **prismoid** is a solid whose ends are *any* parallel (but not similar) figures of the same number of sides, and the other faces are trapeziums.

In the diagram, ABCD and *abcd*, the two parallel ends, are also trapeziums.

If the ends were *similar*, the solid would be the frustum of a pyramid. The frustum of a wedge is a prismoid whose ends are rectangular.

A plane drawn parallel to the base through R, the middle point of the thickness PQ, cuts the solid in the **mid-section** XYZV: and each side of the section is respectively half the sum of the corresponding sides of the ends. For instance $XV = \frac{1}{2}(AD + ad)$, $YZ = \frac{1}{2}(BC + bc)$, and the distance between the parallels XV, YZ is half the sum of the distances between the parallels AD, BC and ad, bc.



108. *To find the volume of a prismoid.*

Let E_1 and E_2 denote the areas of the ends, and M the area of the mid-section: then, if the thickness is k, *the volume of the prismoid* $= \frac{k}{6}[E_1 + E_2 + 4M]$.

This formula applies equally to the *frustum of a wedge*, since the latter solid is a particular case of the prismoid.

Example. In front of a rampart there is a foss of which the top and bottom are horizontal rectangles: the top is 80 feet long by 30 feet broad, and the bottom is 60 feet long by 24 feet broad. If the depth is 12 feet, find how many cubic feet of earth have been excavated.

The figure is a prismoid, in this case the frustum of a wedge.

$$E_1 = 80 \times 30 \text{ sq. ft.} = 2400 \text{ sq. ft.}$$

$$E_2 = 60 \times 24 \text{ sq. ft.} = 1440 \text{ sq. ft.}$$

The mid-dimensions are $\frac{1}{2}(80+60)$ and $\frac{1}{2}(30+24)$, that is 70 feet and 27 feet.

$$\therefore M = 70 \times 27 \text{ sq. ft.} = 1890 \text{ sq. ft.}$$

Now *volume of prismoid* $= \frac{k}{6}[E_1 + E_2 + 4M]$

$$= \frac{12}{6}[2400 + 1440 + 7560] \text{ c. ft.}$$

$$= 22800 \text{ c. ft.}$$

EXAMPLES. XIX.

ON THE WEDGE, FRUSTUM OF THE WEDGE, AND PRISMOID.

[*Elementary and Higher Courses.*]

(*The Wedge.*)

1. Find the volume of a wedge standing upon a rectangular base which measures 7 inches by 5 inches, if the height is 3 inches and the length of the edge 4 inches.

2. Find the weight of a brass wedge, whose height is 5 inches and edge 4 inches, the base being a rectangle which measures 8 inches by 6 inches; if one cubic inch of brass weighs 4·63 oz.

3. A wedge-shaped trench is 40 yards long at the top and 8 feet wide; the length of the bottom edge is 32 yards and the depth is 10 feet. How many tons of earth have been excavated? [Given 1 cubic foot of earth weighs $92\frac{1}{2}$ lbs.]

*4. The triangular faces of a wedge are equally inclined to the base, of which the length is 40 inches and the breadth 18 inches. If the height of the wedge is 1 foot and the edge 30 inches, find the slant surface.

*5. The cross-section of a wedge is an equilateral triangle on a side of 20 inches, and the triangular faces are inclined to the base at an angle of 60°. If the base is 40 inches long, find the volume and slant surface. [$\sqrt{3} = 1\cdot73205$.]

*6. The triangular faces of a wedge are equally inclined to the base, and the dimensions are as follows: length of base a feet, breadth b feet, height h feet, and length of edge e feet. Shew that the slant surface is given by the formula

$$\tfrac{1}{2}\{(a+e)\sqrt{4h^2+b^2}+b\sqrt{4h^2+(a-e)^2}\},$$

and the volume by the formula $\dfrac{hb}{6}(2a+e)$

(The Prismoid)

7. Find the volume of a prismoid whose parallel ends are rectangles, the base measuring 16 inches by 10 inches, and the top 9 inches by 6 inches, the thickness being 4 inches.

8. How many tons of earth are removed in excavating a trench of which the top and bottom are rectangles? At the top it is 400 feet long by 18 feet wide, and at the bottom it is 350 feet long by 15 feet wide. The bottom is horizontal and the depth 12 feet. [Given 1000 cubic feet of earth weigh 42 tons.]

9. The top and bottom of a reservoir are rectangles: the top is 150 feet long by 80 feet broad, and the bottom 120 feet long by 70 feet broad. If the depth is 24 feet, for how long would the reservoir afford a daily supply of 2500 gallons? [1 cubic foot $= 6\frac{1}{4}$ gallons.]

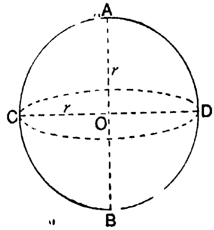
*10. A prismoid stands on a square base, and one pair of opposite slant faces are inclined to the base at an angle of 60°, the other pair at 30°: if the side of the base is 16 feet and the thickness $2\sqrt{3}$ feet, find the dimensions of the top, the volume, and the slant surface.

CHAPTER XX.

THE SPHERE.

109. Definition. A **sphere** is a solid described by the revolution of a semicircle about its diameter, which remains fixed.

Here the semicircle ACB, by revolving about its diameter AB, describes the sphere represented in the figure.



O, the middle point of AB, is the **centre.** Any line which passes through the centre, and is terminated both ways by the surface, is a **diameter**: and any line drawn from the centre to the surface is a **radius.** Thus all radii are equal.

A sphere may also be defined as a solid contained by one curved surface, which is such that all points upon it are equidistant from a fixed point within it, called the centre.

If *any* plane cuts a sphere, the line of section is a circle.

If the cutting plane passes through the centre, the section is called a *central* section or great circle. The area of a plane central section of a sphere is evidently πr^2. From the formula (i) of the next Article we shall see that *the surface of a sphere is four times the area of its plane central section.*

110. *The following formulæ give the surface and volume of a sphere in terms of its radius* r.

(i) *Surface of sphere* $= 4\pi r^2$ square units.

(ii) *Volume of sphere* $= \frac{4}{3}\pi r^3$ cubic units.

Hence if the surface (S) or the volume (V) is given, the radius may be found. For since

(i) $S=4\pi r^2$ and (ii) $V=\frac{4}{3}\pi r^3$,

we have $r=\sqrt{\frac{S}{4\pi}}$, and $r=\sqrt[3]{\frac{3V}{4\pi}}$.

Example i. Find the surface and volume of a sphere whose radius is 3·7 inches, giving a result true to the nearest tenth of a square and cubic inch. $\left(\pi=\frac{22}{7}.\right)$

(i) Surface $=4\pi r^2$

$=4\times\frac{22}{7}\times(3\cdot7)^2=4\times\frac{22}{7}\times13\cdot69$ sq. in.

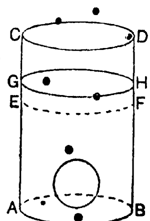
$=172\cdot1$ sq. in.

(ii) Volume $=\frac{4}{3}\pi r^3=\frac{4}{3}\times\frac{22}{7}\times(3\cdot7)^3$

$=\frac{4}{3}\times\frac{22}{7}\times50\cdot65$ c. in. $=212\cdot2$ c. in.

$$\begin{array}{r}
3\cdot7\\
3\cdot7\\
\hline
11\cdot1\\
2\cdot59\\
\hline
r^2=13\cdot69\\
3\ 7\\
\hline
41\cdot07\\
9\ 58\mid 3\\
\hline
r^3=50\cdot65
\end{array}$$

Example ii. An iron sphere of diameter 6 inches is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 1 foot. If the sphere is completely submerged, by how much will the surface of the water be raised?



Here ABDC represents the cylindrical vessel, EF the surface of the water *before* the sphere was dropped in, and GH the surface afterwards. Let $EG=x$ inches.

Then the volume of the sphere equals the volume of the water displaced, that is, the volume of the cylindrical slice EFHG.

(i) Volume of cylinder EFHG

$=\pi\times6^2\times x$ cubic inches.

(ii) Volume of the sphere

$=\frac{4}{3}\pi\times3^3$ cubic inches.

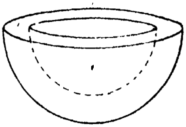
$\therefore\ \pi\times6^2\times x=\frac{4}{3}\pi\,.\,3^3.$

Hence $x=1$ inch.

Example iii. Find the whole surface and weight of a hemispherical copper bowl 1 foot in external diameter and 1 inch thick; having given that 1 cubic inch of copper weighs 5·15 oz. ($\pi = 3·1416$.)

(i) The external curved surface

$$= \frac{1}{2} \times 4\pi \times 6^2 \text{ sq. in.} = 72\pi \text{ sq. in.}$$



The internal curved surface

$$= \frac{1}{2} \times 4\pi \times 5^2 \text{ sq. in.} = 50\pi \text{ sq. in.}$$

The flat circular rim $= \pi(6^2 - 5^2) = 11\pi$ sq. in.

$\therefore$ the total surface $= 72\pi + 50\pi + 11\pi$ sq. in.

$$= 133 \times \pi \text{ sq. in.} = 417·83 \text{ sq. in.}$$

(ii) The volume of the *hemispherical* bowl

$$= \frac{2}{3}\pi(6^3 - 5^3) \text{ cubic inches} = \frac{182}{3}\pi \text{ cubic inches.}$$

$$\therefore \text{ the weight of bowl} = \frac{182}{3}\pi \times 5·15 \text{ oz.}$$

$$= 190·590 \times 5·15 \text{ oz.}$$

$$= 981·5 \text{ oz. nearly.}$$

Obs. As in many other examples the numerical labour is greatly diminished by not substituting the value of π until the final stage.

**Example* iv. The weight of a spherical shell of iron is 1548·025 oz., and its external diameter is 10 inches. Find its thickness, having given 1 cubic inch of iron weighs 4·5 oz.

The volume of iron in the shell $= 1548·025 \div 4·5$ cubic inches

$= 344·005$ cubic inches.

Suppose the thickness to be k inches.

Then $\frac{4}{3}\pi\{5^3 - (5-k)^3\} = 344·005$,

$$\therefore\ 125 - (5-k)^3 = \frac{258·004}{\pi} = 82·125$$

$$(5-k)^3 = 125 - 82·125 = 42·875.$$

Hence $5 - k = \sqrt[3]{42·875}$,

$= 3·5$ nearly,

$\therefore\ k = 1·5$ inch.

	42·875	3 5
	27	
$(30)^2 \times 3 = 2700$	15 875	
$30 \times 5 \times 3 = 450$		
$5^2 = 25$		
3175	15 875	

EXAMPLES. XX. A.

ON SPHERES.

[Elementary Course.]

(Surfaces.)

In the following Examples take $\pi = \frac{22}{7}$.

1. Find the surfaces of the spheres whose radii are respectively

   (i) 7 inches. (Result in square inches.)

   (ii) 1 ft. 9 in. (Result in square feet.)

   (iii) 4 yds. 2 ft. (Result in square yards and feet.)

2. Find the cost of gilding a hemispherical dome 42 feet in diameter, at the rate of 1*s.* 6*d.* per square yard.

3. A hollow cylinder is closed at the ends by hemispheres. If the length of the cylinder is 8 feet, and its diameter 6 feet, find the whole external surface in square feet.

4. A boiler has the form of a hollow cylinder closed at the ends by hemispheres. If the whole length of the boiler is 7 feet, and its diameter is 2 ft. 6 in., find its whole external surface.

5. What is the radius of the sphere whose surface is

   (i) 154 square inches;

   (ii) 17 sq. ft. 16 sq. in.?

6. Find the radius of a sphere, whose surface is equal to the whole surface of a cylinder of height 16 inches, and diameter 4 inches.

(Volumes.)

7. Find the volumes of the spheres whose radii are respectively

   (i) 7 inches. (Result in cubic inches.)

   (ii) 1 ft. 9 in. (Result in cubic feet.)

   (iii) 2·1 inches. (Result in cubic inches.)

8. Find the weight of a solid sphere of lead 6 inches in diameter: having given 1 cubic inch of lead weighs $6\frac{1}{2}$ oz.

9. Find the contents in gallons of a hollow sphere whose internal diameter is 2 feet; having given 1 cubic foot $= 6\frac{1}{4}$ gallons.

10. Find the contents in gallons of a hemispherical vessel, whose internal diameter is 2·8 feet; having given 1 cubic foot $= 6\frac{1}{4}$ gallons.

11. How many spherical bullets, each 1 inch in diameter, could be moulded from a rectangular block of lead 11 inches long, 8 inches wide, and 5 inches thick?

12. How many solid spheres, each 6 inches in diameter, could be moulded from a solid metal cylinder whose length is 45 inches and diameter 4 inches?

13. A piece of lead piping a yard in length is melted down to form spherical shot $\frac{1}{4}$ inch in diameter. If the external diameter of the pipe is 1 inch, and the thickness of the lead $\frac{1}{4}$ inch, how many shot may be made?

14. If a solid cylinder of lead 8 inches long and 6 inches in diameter weighs 92 lbs., find the weight of a leaden sphere one foot in diameter.

(Spherical Shells.)

15. Find, in cubic inches, the volume of a spherical shell, whose internal and external radii are respectively 6 inches and 5 inches.

16. Find, in cubic inches, the volume of a spherical shell whose external diameter is 9 inches, and whose thickness is $1\frac{1}{2}$ inch.

17. Find, in square inches, the whole surface of a hemispherical bowl, one inch thick, and 10 inches in external diameter.

18. Find, in square feet, the whole surface of a hemispherical bowl ·7 inch thick, and 4·2 inches in external diameter.

19. Find (to the nearest ounce) the weight of a spherical shell of copper, having an external diameter of one foot, and a thickness of one inch; having given 1 cubic inch of copper weighs 5·1 oz. nearly.

20. A solid sphere of metal 4 inches in diameter weighs 9 lbs.; find the weight of a spherical shell of the same metal, the external diameter being 20 inches, and the thickness 2 inches.

21. How many pounds of powder would be needed to fill a spherical shell whose internal diameter is 24 inches; given 30 cubic inches of powder weigh 1 lb.?

22. An iron spherical shell, 8 inches in external diameter, and $1\frac{1}{2}$ inch thick, is filled with water. Find (to the nearest ounce) its total weight; having given 1 cubic inch of iron weighs $4\frac{1}{2}$ oz., and 1 cubic inch of water weighs ·6 oz.

(Miscellaneous.)

23. Find the radii of the spheres whose volumes are

(i) $113\frac{1}{7}$ cubic inches;

(ii) $179\frac{2}{3}$ cubic feet.

24. Find the volumes of the spheres whose surfaces are

(i) 616 square inches;

(ii) $38\frac{1}{2}$ square feet.

25. Find the surfaces of the spheres whose volumes are

(i) $179\frac{2}{3}$ cubic inches;

(ii) $4\frac{4}{21}$ cubic feet.

26. A metal sphere, 14 inches in diameter, is dropped into a rectangular cistern, whose base measures 49 inches by $14\frac{2}{3}$ inches. If the sphere is totally submerged, by how much will the surface of the water be raised?

27. A metal sphere, of diameter one foot, is dropped into a cylindrical well, which is partly filled with water. The diameter of the well is 4 feet. If the sphere is completely submerged, by how much will the surface of the water be raised?

EXAMPLES. XX. B.

ON SPHERES.

[Higher Course.]

In the following Examples take $\pi = 3{\cdot}1416$.

(Surfaces.)

1. Find (to the nearest tenth of a square inch) the surface of a solid figure consisting of a cone and hemisphere having the same circular base; the height of the cone is 35 inches, and the diameter of its base 2 feet.

2. If the earth is represented by a globe whose diameter is 3 feet, find (to the nearest hundredth of a square inch) the area enclosed on the globe between two meridians of longitude separated by 1°.

3. Find the ratio of the surface of a sphere to the surface (i) of its circumscribed cylinder, (ii) of its circumscribed cube.

4. Find in square inches (to two places of decimals) the surface of the greatest sphere that can be cut from a solid cone whose vertical angle is 60°, and diameter of base 10 inches.

5. Find the surface of the greatest sphere that can be cut from a cone whose vertical angle is 120°, and diameter of base 1 foot.

6. The external diameter of a hemispherical bowl is 1 ft. 10 in.; find its thickness, if its *whole* surface is 1454·5608 square inches.

(Volumes.)

7. If a solid copper sphere 3 inches in diameter weighs $5\frac{1}{4}$ lbs., find the weight of a sphere of lead $1\frac{1}{2}$ inch in diameter, given that the weights of equal volumes of copper and lead are as 11 : 14.

8. Find the volume of a sphere, whose surface is equal to that of a circle on a diameter of 3 ft. 4 in.

9. A cube and a sphere have equal surfaces: shew that the ratio of their volumes is 72 : 100 nearly.

10. A leaden pipe, 6 centimetres in external diameter, and 1 centimetre thick, is melted down into a solid sphere 20 centimetres in diameter. Find to the nearest millimetre the length of the pipe.

11. A sphere of metal, whose diameter is $\frac{5}{8}$ of an inch, weighs ·625 lb. What is the diameter of a sphere of the same metal which weighs 40 lbs.?

12. If ice loses 7 per cent. of its volume on being melted, find how many gallons of water could be obtained from a sphere of ice 18 inches in diameter, supposing that one cubic foot of water contains $6\frac{1}{4}$ gallons.

13. If a cone of lead, 24 centimetres in height, can be hammered into a solid sphere 8 centimetres in diameter, find to the nearest millimetre the radius of the base of the cone.

14. A closed vessel whose whole external length is 3 ft. 8 in. consists of a cylindrical shell closed at each end by a hemispherical shell of the same diameter and thickness. If the external diameter of this vessel is 1 ft. 8 in., and its thickness 2 inches, find (i) its whole external surface, to the nearest tenth of a square inch, and (ii) its capacity to the nearest gallon, having given 1 cubic foot contains $6\frac{1}{4}$ gallons.

15. From a cubical block of wood the greatest possible sphere is turned. What decimal of the original solid has been cut away?

16. From a cylinder, whose height is equal to its diameter, the greatest possible sphere is turned. What fraction of the original solid has been cut away?

(The following Examples involve the extraction of Cube Root.)

17. A solid metal cylinder is 16 inches high, and 2 feet in diameter. Find the radius of a sphere of the same material and weight.

18. The greatest possible sphere is turned from a cubical block of deal. If the weight of the wood removed is 217·2384 lbs., find the diameter of the sphere, having given 1 cubic foot of deal weighs 57 lbs.

19. A cubical block of metal, each edge of which is 1 foot, is melted down into a sphere. Find its diameter, to the nearest tenth of an inch.

20. A piece of leaden pipe, 8 centimetres in external diameter and 1·5 centimetre thick, is melted down into a solid sphere. If the length of the pipe is 1 metre, find to the nearest millimetre the diameter of the sphere.

21. Find to the nearest tenth of an inch, the internal diameter of a hemispherical vessel capable of containing 11 gallons. [One gallon = 277·25 cubic inches.]

22. If the weights of equal volumes of gold and silver are in the ratio of $19\frac{1}{4}$ to $10\frac{1}{2}$, find the radius of a silver sphere whose weight is equal to that of a sphere of gold of radius 6 inches.

(Spherical Shells.)

23. Find, to the nearest tenth of an inch, the thickness of a spherical shell 14 inches in external diameter, if its weight is equal to that of a solid sphere of the same material, having a diameter of 10·2 inches.

24. A leaden spherical shell, 18 centimetres in external diameter, is melted down into a solid cylinder 8 centimetres high and 12 centimetres in diameter. What was the thickness of the shell?

25. A copper spherical shell 14 inches in external diameter, weighs 407 lbs. Find its thickness, to the nearest tenth of an inch, having given 1 cubic foot of copper weighs 555 lbs. $(\pi = \frac{22}{7})$.

26. A spherical iron shell is 32 inches in external diameter. Find the thickness (to the nearest hundredth of an inch) if the shell is capable of containing 50 gallons. [Given 1 cubic foot = $6\frac{1}{4}$ gallons nearly.]

27. A spherical shell (consisting of two hemispherical shells screwed together) fits tightly over a solid sphere of the same material. If the weight of the sphere and shell are equal, find the thickness of the shell, the external diameter being 1 foot.

(*Miscellaneous.*)

28. A cylindrical cup of external diameter $4\frac{1}{2}$ inches, height $6\frac{1}{4}$ inches, and thickness $\frac{1}{4}$ inch weighs 24 oz. 9 dwts.: what would be the weight of a hemispherical lid, of the same material and thickness, to fit the top?

29. A hollow cone, whose angle is 60°, is held with its axis vertical and vertex downward, and water is poured in until the surface stands 8 inches above the vertex. A metal sphere is then dropped in, and the water rises half an inch. Supposing the sphere to be completely submerged, find its radius to the nearest tenth of an inch.

30. Water is poured into a hollow cone, whose angle is 120°, until the surface stands 10 centimetres above the vertex. An iron spherical shell, of external diameter 8 centimetres, having a small aperture, is then dropped in, so that the water enters the shell and completely submerges it. If the surface of the water is found to have risen 2 millimetres, find the thickness of the shell.

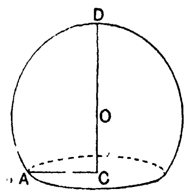
31. A spherical shell, having a small aperture, is placed in a cylindrical vessel of the same diameter, and water is poured into the vessel until the shell is just submerged, its cavity having been filled. The shell is now removed, and the water contained in it poured back into the cylinder. If a *solid* sphere of the same diameter is now dropped in, the surface of the water is found to stand half an inch above its former level; find the thickness of the shell, its external diameter being 6 inches.

32. A spherical shell, of external diameter 8 centimetres, is placed in a conical vessel whose vertical angle is 60°, and water is poured into the vessel, until having entered the cavity of the shell by a small aperture, it just submerges the outer surface. The shell is now removed, and the water contained in it is poured back into the vessel. If a solid sphere of the same diameter is now dropped in, the surface of the water is found to stand 3 millimetres above its former level: find the thickness of the shell.

CHAPTER XXI.

THE ZONE AND SEGMENT OF A SPHERE.

111. DEFINITIONS. 1. A **segment of a sphere** is a part cut off from the sphere by a plane.

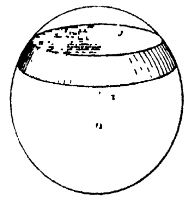


It is clear that any plane, which does not pass through the centre, divides a sphere into two segments, one greater and one less than a hemisphere.

The base of the segment is a circle (see p. 172); and the line CD, drawn at right angles to the base from its centre, and terminated by the curved surface, is called the **height** of the segment.

The centre of the sphere lies in CD, or CD produced towards C.

2. A **Zone of a sphere** is the part cut off from the sphere between two parallel planes.



Thus the zone of a sphere has two circular ends and a curved surface. The **thickness** of the zone is the perpendicular distance between the two cutting planes.

NOTE. Suppose one of the cutting planes to recede from the centre of the sphere, it will be seen that the corresponding end of the zone will be diminished in area: and finally, when the plane *touches* (instead of *cuts*) the sphere, the circular end will vanish altogether. The zone then becomes a segment. Thus a segment of a sphere may be regarded as a special kind of zone and we shall find accordingly that the formulæ for the volume and curved surface of a segment may be derived from the corresponding formulæ for the zone.

112. *To find the curved surface and volume of the segment of a sphere.*

Let the radius of the *sphere* = r units, the radius of the *base* of the segment r_1, and the height h.

(i) *Curved surface of segment* = $2\pi rh$ square units.

(ii) *Volume of segment* = $\frac{\pi h}{6}\{3r_1^2 + h^2\}$ cubic units.

113. *To find the curved surface and volume of the zone of a sphere.*

Let the radius of the *sphere* = r units, the radii of the two *ends* r_1 and r_2, and the thickness k.

(i) *Curved surface of zone* = $2\pi rk$ square units.

(ii) *Volume of zone* = $\frac{\pi k}{6}\{3(r_1^2 + r_2^2) + k^2\}$ cubic units.

It is important to notice that the curved surface of a zone depends only on the radius of the *sphere* and the thickness of the zone. Hence *all zones cut from the same sphere have equal curved surfaces, so long as they are of the same thickness.*

For,

curved surface of zone = (*circumference of* sphere) × (*thickness of zone*).

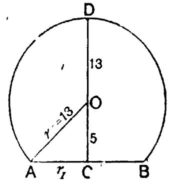
NOTE. Suppose one circular end of a zone to recede from the other end until the zone becomes a segment, then the thickness (k) of the zone becomes the height (h) of the segment; and the formulæ for the curved surfaces become identical.

Again, it has been seen that if one circular end of a zone recedes from the centre, its area is gradually diminished, and ultimately vanishes altogether. Hence when the zone becomes a segment, $r_2 = 0$. Accordingly if we put $r_2 = 0$ in the formula for the volume of a zone

$$\frac{\pi k}{6}\{3(r_1^2 + r_2^2) + h^2\}, \text{ we obtain } \frac{\pi h}{6}\{3r_1^2 + h^2\};$$

the formula for the volume of a segment.

Example i. Find the whole surface and volume of a segment (greater than a hemisphere), its height being 18 inches, and the radius of the sphere from which it was cut being 13 inches.



The figure represents a central section of the segment.

$r = OA = OD = 13$ inches;

and $h = CD = 18$ inches.

$\therefore$ $OC = 5$ inches.

Required r_1, or AC.

From the right-angled triangle OCA,

$r_1^2 = 13^2 - 5^2 = 144$

$\therefore$ $r_1 = 12$ inches.

(i) *Curved* surface $= 2\pi rh$; and area of *base* $= \pi r_1^2$.

$\therefore$ *whole* surface $= 2\pi rh + \pi r_1^2 = \pi(2rh + r_1^2)$

$= \frac{22}{7}(26 \times 18 + 12^2)$ sq. in.

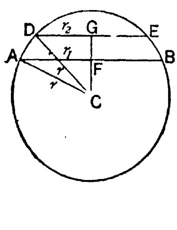
$= 1923\frac{3}{7}$ sq. in.

(ii) Volume $= \frac{\pi h}{6}\{3r_1^2 + h^2\}$

$= \frac{22}{7} \times \frac{18}{6}\{3 \times 12^2 + 18^2\}$ cubic inches

$= 4\frac{1}{8}$ cubic feet.

Example ii. Find the whole surface and volume of a zone, the diameter of the sphere being 1 ft. 8 in., and the distance of the plane ends from the centre (on the same side) 6 inches and 8 inches.



In the figure $r = AC = DC = 10$ inches;

$\therefore$ $CG = 8$ inches, $CF = 6$ inches;

$k = FG = 2$ inches.

Required r_1 and r_2.

From the right-angled triangles AFC, DGC,

$r_1^2 = r^2 - FC^2$; and $r_2^2 = r^2 - GC^2$.

Hence $r_1 = 8$ inches, and $r_2 = 6$ inches.

(i) The *curved* surface and the areas of the two ends are respectively

$2\pi rk$, πr_1^2 and πr_2^2;

$\therefore$ *whole surface* $= \pi\{2rk + r_1^2 + r_2^2\}$

$= \frac{22}{7}\{2 \times 10 \times 2 + 8^2 + 6^2\}$ sq. in. $= 440$ sq. in.

∴ (ii) The volume $=\frac{\pi h}{6}\{3(r_1^2+r_2^2)+h^2\}$

$$=\frac{22}{7}\times\frac{2}{6}\{3(8^2+6^2)+2^2\}\text{ c. in.}=318\cdot17\text{ c.}$$

Example iii. Find the curved surface of a zone of a sphere, if the radii of the two ends are 12 inches and 5 inches, and the thickness 7 inches.

Here $r_1=12$ inches; $r_2=5$ inches, and $h=7$ inches

Required r. Let CF $=x$.

Then $AC^2=r_1^2+x^2$, and $DC^2=r_2^2+(7+x)^2$;

$$12^2+x^2=5^2+49+14x+x^2,$$

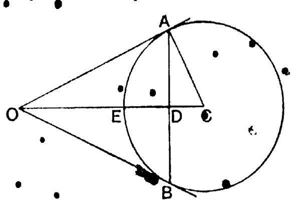
$$x=5\text{ inches.}$$

Hence $r=\sqrt{r_1^2+x^2}=\sqrt{12^2+5^2}=13$ inches;

and curved surface $=2\pi rh=2\times\frac{22}{7}\times13\times7$ sq. in.

$$=572\text{ sq. in.}$$

Example iv. A sphere of diameter 24 feet is placed so that its centre is 37 feet distant from an observer's eye. Find the area of that part of the sphere's surface that is visible to the observer.



Taking a central section of the figure, we see that the *tangents* OA, OB indicate that part of the sphere which is visible from O.

Now OC $=37$ ft.,

and AC $=12$ ft.

$$\therefore OA=\sqrt{OC^2-AC^2}$$

$$=\sqrt{37^2-12^2}=35\text{ ft.}$$

It is required to find DC, and hence ED the height of the segment.

From the similar triangles DCA, ACO,

$$\frac{DC}{CA}=\frac{AC}{CO};\text{ or, }\frac{DC}{12}=\frac{12}{37}.$$

Hence $DC=\frac{144}{37}=3\cdot89$ feet;

∴ h (or ED) $=$ EC $-$ DC $=12-3\cdot89=8\cdot11$ feet nearly;

and curved surface of segment $=2\pi rh=611\cdot7$ sq. ft.

EXAMPLES. XXI. A.

On Zones and Segments of a Sphere.

[*Elementary Course.*]

(*Surfaces.*)

In the following Examples take $\pi = \frac{22}{7}$.

1. Find the *curved* surface of the segment of a sphere

(i) the radius of the sphere being 5 inches, and the height of the segment $3\frac{1}{2}$ inches,

(ii) the diameter of the sphere being 1 yard, and the height of the segment 1 ft. 2 in.

2. Find the *curved* surface of a zone of a sphere

(i) the radius of the sphere being $10\frac{1}{2}$ inches, and the thickness of the zone $1\frac{5}{8}$ inch,

(ii) the diameter of the sphere being 2 ft. 6 in. and the distance between the parallel faces of the zone 4·2 inches.

3. From a sphere 8 inches in diameter, a zone is cut by parallel planes whose distances from the centre on opposite sides are $1\frac{1}{4}$ inch and $\frac{1}{2}$ inch; find its curved surface.

4. Find the *whole* surface of a segment of a sphere

(i) the diameter of the sphere being 10 inches, and the height of the segment 2 inches,

(ii) the diameter of the sphere being 2 ft. 2 in., and the height of the segment 8 inches.

5. Find the whole surface of a zone of a sphere, the diameter of the sphere being 5 ft. 10 in., and the distance of the plane ends of the zone from the centre 1 ft. 9 in. and 2 ft. 4 in. on the same side.

6. From a sphere of diameter 4 ft. 2 in. a zone is cut by parallel planes, whose distances from the centre are 1 ft. 3 in. and 7 inches on opposite sides. Find the whole surface of the zone to the nearest hundredth of a square foot.

7. A copper boiler, whose extreme length is 4 ft. 4 in., consists of a hollow cylinder with ends closed by segments of a sphere. If the cylindrical part is 1 yard in length, and 2 feet in diameter, find (to the nearest square inch) the whole surface of the boiler.

(*Volumes.*)

8. Find the volumes of the segments of a sphere in which

(i) the radius of the base of the segment is $1\frac{1}{2}$ inch, and the height $3\frac{1}{2}$ inches,

(ii) the diameter of the base of the segment is 8 inches, and the height is 6 inches.

9. Find the volumes of the zones of a sphere in which

(i) the radii of the circular ends are 4 inches and 3 inches, and the thickness 3 inches,

(ii) the diameters of the circular ends are 2 feet and 10 inches, and the distance between them is 3 inches.

10. Find the volumes of the segments of a sphere in which

(i) the radius of the sphere is 5 inches, and the radius of the base 4 inches, the segment being greater than a hemisphere,

(ii) the diameters of the sphere and base are 2 ft. 10 in. and 2 ft. 6 in. respectively, the segment being less than a hemisphere.

11. From a sphere of radius 5 inches a zone is cut by parallel planes, whose distances from the centre on opposite sides are 4 inches and 3 inches. Find the volume of the segment.

12. Find the volume of a zone cut from a sphere of diameter 2 ft. 10 in. by parallel planes, whose distances from the centre on the same side are 15 inches and 8 inches respectively.

13. A sphere of diameter 34 inches is cut through by a plane drawn 8 inches from the centre; find the ratio of the volumes of the two segments.

EXAMPLES. XXI. B.

ON ZONES AND SEGMENTS OF A SPHERE.

[*Higher Course.*]

(*Surfaces.*)

In the following Examples, unless otherwise stated, take $\pi = 3{\cdot}1416$.

1. Find the curved surfaces of the segments in which

(i) the radius of the *base* is 4 inches, and the height 2 inches,

(ii) the diameter of the *base* is 5 ft. 10 in., and the height 2 ft. 1 in.

And find the curved surfaces of the zones in which

(iii) the radii of the two *ends* are 4 inches and 3 inches, and the thickness 1 inch,

(iv) the diameters of the *ends* are 15 inches and 7 inches, and the thickness 2 inches.

2. A sphere of diameter 80 feet is placed so that its centre is 50 feet distant from an observer's eye. Find (to the nearest square foot) the area of that part of the sphere's surface that is visible to the observer.

3. A sphere of diameter 16 inches, is placed so that its centre is 17 inches distant from an observer's eye; find what decimal of the whole surface of the sphere is that part which is visible to the observer.

4. How far distant must the centre of a sphere of radius 1 foot be placed from an observer's eye, in order that $\frac{7}{18}$ of the whole surface may be visible to him?

5. Considering the Earth as a sphere of diameter 8000 miles, at what height above the ground would one-millionth of the earth's surface be visible? [Give the result to the nearest foot.]

(*Volumes.*)

6. A vertical cylindrical vessel, whose internal diameter is 4 feet, is completely filled with water. If a metal sphere of radius 25 inches is now laid upon the rim of the vessel, find to the nearest pound what weight of water will overflow.

7. A conical wine-glass, whose angle is 60°, and whose depth is 4 inches, is completely filled with water. If a metal sphere of diameter $5\frac{1}{3}$ inches, is now placed upon the rim of the glass, what fraction of the whole contents will overflow?

(Miscellaneous.)

8. An iron sphere of radius 5 inches is placed at the bottom of a cylindrical vessel whose internal diameter is 1 foot, and water is poured in to a depth of 1 inch. What fraction of the whole surface of the sphere is immersed? And if the sphere is withdrawn from the vessel, what will be the depth of the water?

9. A metal sphere is placed at the bottom of a cylindrical vessel whose internal diameter is 38 centimetres, and water is poured in to a depth of 1 centimetre. If, when the sphere is withdrawn, the depth of the water becomes $\frac{35}{\text{[illegible]}}$ centimetre, what is the radius of the sphere?

10. If, when a sphere of cork floats in water, the height of the submerged segment is $\frac{3}{4}$ of the radius, shew that the weights of equal volumes of cork and water are as $3^4 : 4^4$. [Note. The weight of a floating body is equal to the weight of the water displaced.]

11. Two tangents OP, OQ are drawn to touch a circle of radius 8 inches at P and Q: the angle between them being 60°. If the figure is made to revolve about the line joining O to the centre of the circle, find the surface and volume of the solid so formed.

12. Two tangents OP, OQ are drawn to touch a circle of radius 10 inches at P and Q, the angle between the tangents being 120°. If the figure is made to revolve about the line joining O to the centre of the circle, find the whole surface of the solid so formed.

13. A sphere is placed in a paper cone whose vertical angle is 90°, and the paper is cut away along the circle of contact,

(i) what decimal of the whole surface of the sphere is concealed by the cone?

(ii) what decimal of the whole volume of the sphere is that segment which lies within the cone? [$\sqrt{2} = 1.41421$.]

CHAPTER XXII.

SIMILAR SOLIDS. SOLID RINGS.

SECTION I.

SIMILAR SOLIDS.

114. **Similar solids** may be described as those which have the same *shape*, but not necessarily the same size.

All cubes are similar solids.

All spheres are similar.

Right Prisms are similar, when their bases are similar, and their heights are proportional to corresponding sides of the bases. The same test of similarity applies to *right pyramids*.

Right Cylinders are similar, when their heights are proportional to the radii of their bases: and *similar cones* are known by the same test.

Or again, right circular cones are similar if they have equal vertical angles.

115. The following rules apply to all similar solids.

(i) *The surfaces of similar solids are proportional to the squares of corresponding edges, or of any corresponding lines that may be drawn in them.*

(ii) *The volumes of similar solids are proportional to the cubes of corresponding edges, or of any corresponding lines that may be drawn in them.*

For example, the surfaces of similar cones are proportional to the squares of their heights; and the volumes to the cubes of the heights.

Or again, if the diameter of one sphere is three times that of another, then the surface of the first is 9 times the surface of the second; and the volume of the first sphere is 27 times that of the second.

Example i. If the diameter of the Earth is taken as 7900 miles, and that of the Moon as 2160 miles, how many times greater is the Earth than the Moon? And how many times greater is the surface of the Earth than the surface of the Moon?

(i) Now volume of Earth : volume of Moon $=(7900)^3 : (2160)^3$.

$$\therefore \frac{\text{Volume of Earth}}{\text{Volume of Moon}}=\left(\frac{7900}{2160}\right)^3=(3\cdot66\ldots)^3=49\cdot02\ldots$$

Thus the Earth is nearly 49 times greater than the Moon.

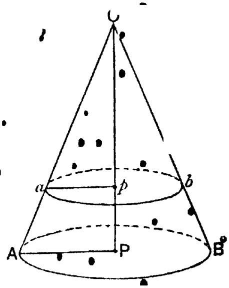
(ii) Again, Surface of Earth : Surface of Moon

$$=(7900)^2 : (2160)^2,$$

$$\therefore \frac{\text{Surface of Earth}}{\text{Surface of Moon}}=\left(\frac{7900}{2160}\right)^2=(3\cdot66\ldots)^2=13\cdot39.$$

Thus the surface of the Earth is rather more than 13 times that of the Moon.

Example ii. What must be the thickness of the frustum cut from a cone 100 inches in height, if the volume of the frustum is $\frac{1}{10}$ that of the cone?



The cones Oab, OAB are similar: and if the frustum AabB is $\frac{1}{10}$ of the original cone, then the cone Oab is $\frac{9}{10}$ of the cone OAB.

$$\text{Or,} \quad \frac{\text{cone } Oab}{\text{cone } OAB}=\frac{9}{10}.$$

$$\text{But} \quad \frac{\text{cone } Oab}{\text{cone } OAB}=\frac{Op^3}{OP^3};$$

$$\therefore \frac{Op^3}{OP^3}=\frac{9}{10}; \text{ or } Op=OP\times\sqrt[3]{\cdot9}$$

$$=100\times\cdot965=96\cdot5 \text{ inches.}$$

Hence the thickness $Pp=OP-Op=3\cdot5$ inches.

EXAMPLES. XXII. A

ON SIMILAR SOLIDS.

1. The edges of two cubes are as 4 : 3, find the ratio of their surfaces and of their volumes.

2. The surfaces of two spheres are in the ratio of 25 : 4; find the ratio of their volumes.

3. The volumes of two spheres are as 343 : 64; express the surface of the first as a decimal of the surface of the second

4. The weights of two similar cones of the same substance are as 1331 : 729; find the ratio of the radii of their bases.

5. The weights of two similar cylinders of the same substance are 13824 lbs. and 12167 lbs.; if the height of the first is 16 feet, find the height of the other.

6. The diameters of two spheres of different substances are as 7 : 3, and the weight of the first is 15 lbs. If the weights of equal volumes of the two substances are as 27 : 49, find the weight of the second sphere.

7. The weights of two spheres are as 9 : 13, and the specific gravities of their substances are respectively 1·69 and ·81. If the diameter of the first is 45 inches, find the diameter of the other.

8. The weights of two similar cylinders (of different substances) are as 49 : 45; if their diameters are in the ratio 14 : 15, compare the weights of equal volumes of the two substances.

9. A pyramid, whose height is 16 inches, and whose base is a square on a side of 14 inches, is intersected by a plane parallel to the base and 1 foot distant from it. Find the volume of the pyramid cut off.

10. At what distance from the base must a cone, whose height is 1 foot, be cut by a plane parallel to the base in order to be divided into two parts of equal volume?

11. A pyramid, 18 inches in height, stands on a square base, whose side is 9 inches. At what distance from the base must it be intersected by a parallel plane, if the volume of the pyramid cut off is 16 cubic inches?

12. In what ratio must the height of a cone be intersected by a plane parallel to the base, if the volume of the frustum cut off is $\frac{1}{n}$th of the whole cone?

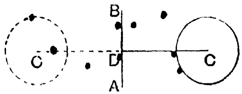
13. A right circular cone is cut by a plane parallel to the base through the middle point of the height. Compare the volumes of the two parts into which the cone is divided.

14. A right circular cone is intersected by two planes parallel to the base and trisecting the height. Compare the volumes of the three parts into which the cone is divided.

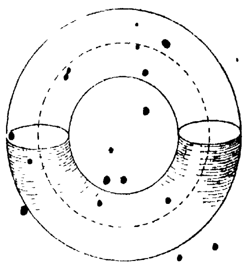
SECTION II.

SOLID CIRCULAR RINGS.

116. Suppose a circle whose centre is C to be connected by a thin bar CD with an axis AB. If the bar revolves about the axis, the solid figure described is called a **solid circular ring**.



The original circle is the *cross-section* of the ring: and the dotted circle described by the centre of the revolving circle is called the *length of the ring*. The length of the ring is half the sum of its inner and outer circumferences.



117. *To find the surface and volume of a solid circular ring.*

The surface and volume of a solid ring are equal to those of a cylinder whose base is the cross-section, and whose height is the length of the ring

Hence (i) *Surface of ring*

$= (\textit{circumference of cross-section}) \times (\textit{length of ring}).$

(ii) *Volume of ring* $= (\textit{area of cross-section}) \times (\textit{length of ring}).$

Example. The outer and inner diameters of a solid ring are 17·5 inches and 10·5 inches; find its surface and volume. $\left(\pi = \frac{22}{7}.\right)$

Here the mean diameter $= \frac{1}{2}(17{\cdot}5 + 10{\cdot}5) = 14$ inches

$\therefore$ Length of ring $= 14\pi$ inches.

And the radius of cross-section $= \frac{1}{4}(17{\cdot}5 - 10{\cdot}5) = 1{\cdot}75$ inches.

$\therefore$ circumference of cross-section $= \frac{7}{2} \times \pi$ inches,

and area of cross-section $= \pi\,(1{\cdot}75)^2$ sq. inches.

Now (i) *Surface of ring* = (*circumference of cross-section*) $\times$ (*length of ring*)

$$= \frac{7}{2}\pi \times 14\pi \text{ sq. in.}$$

$$= 484 \text{ sq. in.}$$

And (ii) *volume of ring* = (*area of cross-section*) $\times$ (*length of ring*)

$$= \pi\,(1{\cdot}75)^2 \times 14\pi \text{ cubic inches.}$$

$$= 423{\cdot}5 \text{ cubic inches.}$$

EXAMPLES. XXII. B.

SOLID CIRCULAR RINGS.

[*In the following Examples take* $\pi = \frac{22}{7}$.]

1. Find the surfaces and volumes of the solid rings in which

(i) the radius of the inner circumference is $10\frac{1}{2}$ inches, and the diameter of the cross-section $3\frac{1}{2}$ inches,

(ii) the diameter of the outer circumference is $6\frac{1}{8}$ inches, and the radius of the cross-section $\frac{7}{8}$ inch,

(iii) the diameters of the outer and inner circumferences are $12{\cdot}6$ inches, and $9{\cdot}8$ inches respectively.

2. The volume of a solid ring is $81{\cdot}312$ cubic inches, and the diameter of its cross-section is $1{\cdot}4$ inch; find the length of the ring, and its inner and outer diameters.

3. The mean diameter of a solid ring is $1{\cdot}75$ inch, and its volume is $\cdot3388$ cubic inch, find the diameter of its cross-section.

4. A solid circular ring fits closely round a cylinder of the same diameter ($2r$) as the cross-section of the ring; find the height of the cylinder (i) if their volumes are equal, (ii) if the whole surface of the cylinder is equal to that of the ring.

5. A solid circular ring fits closely round a sphere. If the diameter of the cross-section of the ring is equal to the radius of the sphere, what ratio does (i) the surface of the ring bear to the surface of the sphere, (ii) the volume of the ring to the volume of the sphere?

CHAPTER XXIII.

LOGARITHMIC TABLES APPLIED TO MENSURATION.

118. The present chapter is designed to shew how the numerical labour involved in some problems of Mensuration may be abridged by the use of logarithms; and to illustrate the application of Mathematical Tables to questions in which angles occur other than those whose trigonometrical ratios are readily determined.

Example i. Find the area of a triangular field whose base measures 18 chains 47 links, the angles at the base being 73° 18′ 30″ and 82°.

[Given $\log 2 = \cdot 3010300$; $L \sin 82° = 9 \cdot 9957528$.

$L \sin 73° 18' = 9\cdot 9812850$, diff. for $60'' = 379$;

$L \sin 24° 41' = 9 \cdot 6207634$, diff. for $60'' = 2748$;

$\log 1847 = 3 \cdot 2664669$;

$\log 38731 = 4 \cdot 5880587$, diff. for $1 = 112$.]

Here $a = 18\cdot 47$ chains, $\mathrm{B} = 73° 18' 30''$, $\mathrm{C} = 82°$, $\therefore \mathrm{A} = 24° 41' 30''$.

$$\text{Now } area\ of\ \Delta = \frac{a^2 \sin \mathrm{B} \sin \mathrm{C}}{2 \sin \mathrm{A}}$$

$$= \frac{(18\cdot 47)^2 \cdot \sin 73° 18' 30'' \cdot \sin 82°}{2 \sin 24° 41' 30''} \text{ sq. chains.}$$

$$\begin{array}{ll}
2 \log 18\cdot 47 & = 2\cdot 5329338 \\
\log \sin 73° 18' & = \bar{1}\cdot 9812850 \\
\text{diff. for } 30'' & = \phantom{\bar{1}\cdot 9812}189 \\
\log \sin 82° & = \bar{1}\cdot 9957528 \\
\hline
 & 2\cdot 5099905 \\
 & \bar{1}\cdot 9219308 \\
\hline
\log \Delta & = 2\cdot 5880597 \\
\log 387\cdot 31 & = 2\cdot 5880587 \\
\hline
 & 10
\end{array}
\qquad
\begin{array}{ll}
\log 2 & = \phantom{\bar{1}}\cdot 3010300 \\
\log \sin 24° 41' & = \bar{1}\cdot 6207634 \\
\text{diff. for } 30'' & = \phantom{\bar{1}\cdot 620}1374 \\
\hline
 & \bar{1}\cdot 9219308
\end{array}$$

Now diff. for $\cdot 01 = 112$,

$\therefore$ prop^{l}. increase $= \frac{10}{112} \times \cdot 01 = \cdot 00089$.

Hence $\Delta = 387\cdot 31089$ sq. chains $= 38\cdot 731089$ acres

$= 38$ ac. 2 r. 37 p. nearly.

Example ii. The volume of a cone is 3682·475 cubic inches, and its height is 17·752 inches; find its vertical angle.

[Given $\log 36824 = 4{\cdot}5660130$, diff. for $1 = 118$;
$\log 17752 = 4{\cdot}2492473$; $\log \pi = {\cdot}4971499$;
$L \tan 38° 24' = 9{\cdot}8990487$, diff. for $60'' = 2595$.]

Suppose the height and radius of the base are h and r inches, and the vertical angle $2a$.

Then $$r = h \tan a,$$

and $$\frac{1}{3}\pi r^2 h = 3682{\cdot}475,$$

or $$\frac{1}{3}\pi h^3 \tan^2 a = 3682{\cdot}475,$$

$$\therefore\ \tan^2 a = \frac{3 \times 3682{\cdot}475}{(17{\cdot}752)^3 \times \pi};$$

$$\therefore\ 2 \log \tan a = \log 3 + \log 3682{\cdot}475 - 3 \log 17{\cdot}752 - \log \pi.$$

$\log 3$	$= {\cdot}4771213$	diff. for $\cdot 1 = 118$
$\log 3682{\cdot}4$	$= 3{\cdot}5660130$	$\therefore$ diff. for $\cdot 075 = 89$
add for $\cdot 075$	89	
	$4{\cdot}0431432$	$\log 17{\cdot}752 = 1{\cdot}2492473$
	$1{\cdot}2448918$	3
	$2 \mid \bar{1}{\cdot}7982514$	$3{\cdot}7477419$
$\log \tan a$	$= \bar{1}{\cdot}8991257$	$\log \pi = {\cdot}4971499$
$\log \tan 38° 24'$	$= \bar{1}{\cdot}8990487$	$4{\cdot}2448918$
	770	

$$\text{Required addition} = \frac{770}{2595} \times 60'' = 17''{\cdot}8;$$

$$\therefore\ a = 38° 24' 17''{\cdot}8; \text{ so that } 2a = 76° 48' 35''{\cdot}6.$$

119. In the following example the logarithms and differences are supposed to be taken as required from the Tables.

Example. To the ends of a cylinder, whose length is equal to its diameter, are applied hemispheres of the same diameter as the cylinder. If the volume of the whole figure thus formed is 463·873 cubic inches, find its surface. [$\pi = 3{\cdot}1415926$.]

Suppose the radius of the cylinder and hemispheres to be r inches. Let the volume of the whole figure be **V**, and let the whole surface be **S**.

Then V = volume of cylinder + volume of two hemispheres

$$=\pi r^2\times 2r \quad + \quad \frac{4}{3}\pi r^3$$

$$=\frac{10}{3}\pi r^3 \text{ cubic inches.}$$

Thus $$\frac{10}{3}\pi r^3 = 463\cdot873.$$

$$\therefore\ r=\left\{\frac{3\times 46\cdot3873}{\pi}\right\}^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots\text{(i)}.$$

And S = surface of cylinder + surface of two hemispheres

$$=\quad 2\pi r\times 2r \quad + \quad 4\pi r^2$$

$$= 8\pi r^2 \text{ square inches}$$

$$= 8\pi\left\{\frac{3\times 46\cdot3873}{\pi}\right\}^{\frac{2}{3}} \text{ from (i)}$$

$$= 8\{\pi\times 3^2\times(46\cdot3873)^2\}^{\frac{1}{3}};$$

$$\therefore\ \log S=\log 8+\frac{1}{3}\{\log\pi+2\log 3+2\log 46\cdot3873\}.$$

From the Tables,	log 46·387	=	1·6663963
add for	3		28
			1·6663991
			2
			3·3327982
	2 log 3	=	·9542426
From the Tables,	log 3·1415	=	·4971371
add for	9		124
	2		2·8
	6		[illegible]
		3	4·7841907
			1·5947302
	log 8	=	·9030900
	log S	=	2·4978202
From the Tables,	log 314·64	=	2·4978139
			63
			55
			8
			8·3

$$\therefore\ S = 314\cdot6446 \text{ square inches.}$$

EXAMPLES. XXIII.

MISCELLANEOUS QUESTIONS TO BE WORKED BY LOGARITHMS.

(Areas of Plane Figures.)

[*In Examples* 1—10 *take* $\log 2 = \cdot 30103$; $\log 3 = \cdot 47712$; $\log 7 = \cdot 84510$; $\log \pi = \cdot 49715$.]

1. Find the acreage of a triangular field whose sides are 2500, 3500 links and 4000 links.

[Given $\log 43302 = 4 \cdot 63650$.]

2. In the triangle **ABC**, $a = 24$ yards, $b = 36$ yards, and $c = 40$ yards; find the length of the perpendicular from **C** on **AB**.

[Given $\log 13 = 1 \cdot 11394$; $\log 2 \cdot 1331 = \cdot 32900$.]

3. Find the length of the perpendicular drawn from **A** on **BC** in the triangle **ABC**, if $a = 800$ feet, $b = 671 \cdot 7$ feet, and $c = 528 \cdot 3$ feet.

[Given
$\log 3283 = 3 \cdot 51627$; $\log 4717 = 3 \cdot 67367$; $\log 43998 = 4 \cdot 64343$.]

4. The sides of a triangle are 15 inches, 16 inches, and 17 inches; find the side of an equilateral triangle of equal area, giving a result true to the nearest thousandth of an inch.

[Given $\log 79523 = 4 \cdot 90138$.]

5. Find the area of a triangular field of which two sides measure 8 chains 75 links and 2 chains 45 links, the included angle being $64° 20'$.

[Given $L \sin 64° 20' = 9 \cdot 95488$; $\log 96717 = 4 \cdot 98503$.]

6. Find (to the nearest square inch) the area of a circle whose radius is $71 \cdot 27$ inches.

[Given $\log 7127 = 3 \cdot 85291$; $\log 1 \cdot 5958 = \cdot 20297$; $\log \pi = \cdot 49715$.]

7. A strip of ground has the shape of a trapezium **ABCD**. The parallel sides **AB** and **CD** measure respectively 1076 links and 924 links; the length of **AD** is 64 links, and the angle **BAD** is $71° 13'$. Find the acreage of the ground.

[Given $L \sin 71° 13' = 9 \cdot 97623$; $\log 60591 = 4 \cdot 78241$.]

8. Find the diameter of a circle whose area is equal to that of an equilateral triangle on a side of 18 inches.

[Given $\log 66827 = 4{\cdot}82495$.]

9. The two diagonals of a quadrilateral are inclined to one another at an angle of 28° 15′, and measure respectively 24·3 feet and 25·6 feet. Find its area.

[Given $L \sin 28° 15' = 9{\cdot}67515$; $\log 14722 = 4{\cdot}16796$.]

10. Find the area of a triangle of which the base is 100 feet, and the adjacent angles are 51° 18′ and 42° 12′.

[Given $L \sin 51° 18' = 9{\cdot}89233$;
$L \sin 42° 12' = 9{\cdot}82719$;
$L \sin 86° 30' = 9{\cdot}99919$;
and $\log 26{\cdot}26 = 1{\cdot}41930$.]

(*The results in the following examples are to be given to six significant digits, the seventh digit is to be found, and used if necessary to correct the sixth.*)

11. Find the area of a regular polygon of 16 sides inscribed in a circle whose radius is 100 inches.

[Given $\log 2 = {\cdot}3010300$; $L \sin 22° 30' = 9{\cdot}5828397$;
$\log 30614 = 4{\cdot}4859201$, diff. for 1 = 142.]

12. Find the area of a triangle whose base is 243 yards, the adjacent angles being 59° 8′ and 42° 52′.

[Given
$\log 2 = {\cdot}3010300$, $\log 3 = {\cdot}4771213$; $L \sin 59° 8' = 9{\cdot}9336713$;
$L \sin 42° 52' = 9{\cdot}8326970$; $L \sin 78° = 9{\cdot}9904044$;
$\log 17625 = 4{\cdot}2461291$, diff. for 1 = 247.]

13. The area of a quadrilateral enclosure is 12 acres, and the two diagonals measure 16 chains and 20 chains respectively. At what angle are the diagonals inclined to one another?

[Given $\log 2 = {\cdot}3010300$; $\log 3 = {\cdot}4771213$;
$L \sin 48° 35' = 9{\cdot}8750142$; diff. for 1′ = 1114.]

14. About a square whose side is 1 inch, a circle is circumscribed, and about this circle a square: about the second square a circle, and so on. Find the area of the 18th circle so drawn.

[Given $\log 2 = {\cdot}3010300$; $\log \pi = {\cdot}4971499$;
$\log 20588 = 4{\cdot}3136142$, diff. for 1 = 211.]

15. Supposing the earth to be a true sphere whose diameter is 7925·5 miles, find the length of the arctic circle (lat. 66° 30′).

[Given log 79255 = 4·8990267 ; log π = ·4971499 ;
L sin 23° 30′ = 9·6006997 ;
log 99283 = 4·9968749, diff. for 1 = 44.]

(Surfaces and Volumes of Solids.)

[*In the following Examples take* log 2 = ·3010300 ;
log 3 = ·4771213 ; log π = ·4971499.

Give the results to six significant digits, using the seventh where necessary to correct the sixth.]

16. Find, in inches, the edge of a cubical block of granite weighing 250 tons.

[Given 1 cubic foot of granite weighs 2660 ounces,
log 19 = 1·2787536 ; log 17988 = 4·2549961, diff. for 1 = 243.]

17. Find, to the nearest gallon, the capacity of a cubical cistern, each side of which measures 378·2 inches.

[Given 1 gallon = 277·274 cubic inches, log 3782 = 3·5777215 ;
log 27727 = 4·4429029, diff. for 1 = 157,
log 19509 = 4·2902350, diff. for 1 = 222.]

18. How many feet of wire ·48 inch in diameter may be drawn from 1 ton of copper?

[Given 1 cubic foot of copper weighs 554·87 lbs.,
log 7 = ·8450980 ; log π = ·4971499 ;
log 55487 = 4·7441912 ;
log 32125 = 4·5068431, diff. for 1 = 135.]

19. How many spherical shot, each ·16 inch in diameter, could be made from 704·95 lbs. of lead?

[Given 1 cubic inch of lead weighs 6·574 oz.,
log 6574 = 3·8178297 ; log 70495 = ·8481584.]

20. Find the volume of the greatest cube that could be cut from a sphere whose diameter is 37·625 inches.

[Given log 37625 = 4·5754765 ;
log 10250 = 1·0107239, diff. for 1 = 424.]

21. A cubical block of ice, each edge of which measures 9·3 inches, is placed in an empty cylindrical tank. If the ice on melting loses 7 per cent. of its volume, and the diameter of the tank is 1 yard, find as a decimal of an inch how deep the water will stand in the tank.

[Given log 31 = 1·4913617;

log 73491 = 4·8662342, diff. for 1 = 59.]

22. Find the diameter of a sphere whose volume is equal to that of a right prism 81 inches high, standing on a regular hexagonal base whose side is 54 inches.

[Given log 10543 = 4·0229642, diff. for 1 = 412.]

23. Find, in inches, the diameter of a solid copper sphere equal in weight to a cube of iron whose edge is 1 yard, if the weight of equal volumes of copper and iron are as 4439 : 3894.

[Given log 3894 = 3·5903959; log 4439 = 3·6472851;

log 42756 = 4·6309971, diff. for 1 = 102.]

24. A right pyramid stands on a square base whose side is 247·32 inches, and each of the slant faces is inclined to the base at an angle of 52° 20′. Find the volume.

[Given log 24732 = 4·3932592; *L* tan 52° 20′ = 10·1124058;

log 32661 = 4·5140295, diff. for 1 = 133.]

25. A right pyramid, whose height is 9 inches, stands on a regular hexagonal base, each side of which is 16 inches. At what angle are the slant faces inclined to the base?

[Given log 2 = ·3010300; log 3 = ·4771213;

L tan 33° = 9·8125174, diff. for 60″ = 2765.]

26. Find the volume of a cone whose vertical angle is 8[illegible] 30′, the diameter of the base being 43·68 inches.

[Given log 2184 = 3·3392526; *L* tan 45° 15′ = 10·0113761;

log 11198 = 4·0491405, diff. for 1 = 388.]

27. A right pyramid stands upon a square base, and its triangular faces are equilateral. Find the angle of inclination between a slant face and the base.

[Given *L* tan 54° 44′ = 10·1504784, diff. for 60″ = 2680.]

28. Find the vertical angle of a cone whose volume is 8682·74 cubic inches, if the radius of the base is 12·8 inches.

[Given log 86827 = 4·9386548, diff. for 1 = 50;

L tan 75° 48′ = 10·5968127, diff. for 60″ = 5315.]

29. The water contained in a cubical cistern, each edge of which measures 6 feet, is found to lose by evaporation ·04 of its volume in a day. Assuming the loss to arise from evaporation only, find how many ounces of water will be left in the cistern after the expiration of 10 days.

[Given 1 cubic foot of water weighs 1000 oz.;

log 14360 = 4·1571544, diff. for 1 = 303.]

30. A trench has the following dimensions: width at the top 11 feet, width at the bottom (which is horizontal) 9 feet; length 20 feet, and depth 5 feet. It is filled with water, and every day $\frac{1}{10}$ of the water which remained at the beginning of the day is drawn off. How many gallons will be left after the 8th drawing?

[Given 1 gallon = 277·274 cubic inches,

log 27727 = 4·4429029, diff. for 1 = 157;

log 26827 = 4·4285721, diff. for 1 = 162.]

[*In the following Examples all necessary logarithms and differences are to be taken from the Tables. In solving these questions Chambers' Mathematical Tables have been used. The results have been calculated to seven significant digits, the seventh figure being used to correct the sixth.*]

31. Find the radius of a sphere whose volume is $\frac{5}{17}$ of that of a pyramid of height 153·215 inches, and whose base contains 34·51809 square inches.

32. Find the vertical angle of a cone whose volume is 51904·77 cubic inches, and height 57·3568 inches.

33. Find the circumference of the base of a cone whose volume is 218·9694 cubic inches, and height 14·877 inches.

34. How many cubic feet of air are contained in a tent in the form of a cylinder surmounted by a cone, the radius of the base being 117·8 inches, the height of the *cylindrical* part 124·3 inches, and the extreme height to the vertex of the cone 217·9 inches?

35. The ends of a cylinder, whose length is equal to its diameter, are capped by two hemispheres of the same diameter as the cylinder. If the whole surface of the solid so formed is 462·183 square inches, find the volume.

36. Find the edge of a regular tetrahedron whose volume is 6251·37 cubic inches.

37. A right pyramid stands on a regular hexagonal base whose side is 8 inches, and its volume is 378·627 cubic inches. Find the angle which each slant face makes with the base.

38. A steel cone weighs 286·453 oz.; if the vertical angle is 84° 26′ 20″ find the radius of the base, supposing that 1 cubic inch of steel weighs 4·533 oz.

39. Find the inclination to one another of any two faces of a regular tetrahedron which have a common edge.

40. Find the angle between two adjacent faces (*i.e.* faces which have a common edge) of a regular octahedron.

41. A crystal consists of a cube capped at two opposite ends by pyramids whose faces are equilateral triangles. If the volume of the whole figure thus formed is 8·2715 cubic inches, find the length of each edge.

42. A solid figure consists of a cone having a vertical angle of 60° and a hemisphere on the same base. If its volume is 4682·7 cubic inches, find the radius of the base, and the whole surface.

43. A right pyramid stands on a square base, each side of which is 20 inches: and each slant edge of the pyramid is 15 inches. Find the inclination of each slant face to the base.

44. A cone whose height is 38·26 inches has a vertical angle of 78° 46′ 30″; find its volume and curved surface.

45. The volume of a frustum of a cone is 9529·5 cubic inches. If the diameters of the ends are 12 inches and 10 inches, find the thickness; and determine the vertical angle of the cone from which the frustum was cut.

46. The volume of a frustum of a pyramid standing on a square base is 1734·281 cubic inches. If the thickness is 16·348 inches, and if the area of the base is 9 times the area of the top, find the dimensions of each end.

47. The cross-section of a tunnel, bored through a limestone hill, consists, as to its lower part, of three sides of a regular hexagon **AB**, **BC**, **CD** (the middle side **BC** being horizontal), and the section of the roof is a semicircle described on **AD**. If the length of the tunnel is 120 yards, and its width at the base **BC** is 30 feet, how many tons of stone have been removed in the excavation, given 1 cubic foot of limestone weighs 170 lbs.?

48. The diameter of a sphere is 32·01161 inches, and its volume is 5·6 times as great as that of a cone whose height is 69·6687 inches. Find the radius of the base of the cone.

49. A cylindrical shaft, 4 feet 9 inches in diameter, is sunk into a quartz-reef. If the yield of gold averages 1 oz. 15 dwts. per ton of stone crushed, how much gold will be obtained when the depth of the shaft is 128 feet? [Given 1 cubic foot of quartz weighs 171·875 lbs. Av.]

50. From a sphere of copper of radius 4·769 inches, 5·9217 lbs. of wire ·178 inch in diameter are made; and from the remaining copper a solid cylinder 99·7824 inches long is constructed. Find how many times the wire can be wound round the cylinder, having given that 1 cubic foot of copper weighs 555 lbs.

51. If V denotes the volume of a sphere, and S the area of its surface, prove that

$$3 \log \mathsf{S} = 2 \log 6 + \log \pi + 2 \log \mathsf{V}.$$

And calculate the value of S, if $\mathsf{V} = 796·327$ cubic inches.

52. A square on a side of 19 inches revolves about its diagonal. Find the surface and the volume of the solid figure so described.

53. Find the surface and volume of the solid described by the revolution of a regular hexagon about one of its sides, the length of each side being 20 inches.

CHAPTER XXIV

COLLECTION OF FORMULÆ. EXAMINATION QUESTIONS.

120. We here collect for reference and revision the formulæ most generally used in Elementary Mensuration.

I. Plane Mensuration.

1. Right-angled Triangles. [Chapter II. Page 7.]

   (i) $(Hypotenuse)^2 = a^2 + b^2$.

   (ii) $Diagonal\ of\ square = side \times \sqrt{2}$.

2. Rectangles and Squares. [Chapter III. Page 14.]

   (i) $Area\ of\ rectangle = length \times breadth$.

   (ii) $Area\ of\ square = (side)^2$.

3. Triangles. [Chapter IV. Page 30.]

   (i) $$Area\ of\ any\ triangle = \sqrt{s.(s-a)(s-b)(s-c)}$$
   $$= \frac{1}{2} bc \sin A$$
   $$= \frac{a^2 \sin B \sin C}{2 \sin A}.$$

   (ii) $Area\ of\ right\text{-}angled\ triangle = \frac{1}{2} ab$.

   (iii) $Area\ of\ equilateral\ triangle = m^2\sqrt{3}$ (side $= 2m$).

4. Quadrilaterals. [Chapter V. Page 42.]

   (i) $Area\ of\ parallelogram = base \times height$

(ii) *Area of rhombus* $= \frac{1}{2}$ (*product of diagonals*).

(iii) *Area of trapezium* $= \frac{1}{2}$ (*height*) $\times$ (*sum of parallel sides*).

(iv) *Area of any quadrilateral* $= \frac{1}{2}$ (*diagonal*) $\times$ (*sum of offsets*)

$$= \frac{1}{2} dd' \sin \theta.$$

5. Circles. [Chapter VI. Page 51.]

(i) *Circumference of circle* $= 2\pi r$.

(ii) *Area of circle* $= \pi r^2$.

6. Sectors of Circles. [Chapter VIII. Page 75.]

(i) *Area of sector* $= \frac{1}{2}$ *arc* $\times$ *radius*

$$= \frac{D}{360} \times \textit{area of circle}$$

$$= \frac{1}{2} r^2 \theta.$$

7. Segments of Circles. [Chapter VIII. Page 78.]

(i) *Area of segment* = *sector* − *triangle*.

$$= \frac{1}{2} r^2 (\theta - \sin \theta).$$

8. Regular Polygons. [Chapter X. Page 94.]

(i) *Area of regular polygon* $= \frac{n}{2} \times$ *side* $\times$ (*perp*. *from centre*)

$$= a^2 \times \frac{n}{4} \cot \frac{180^\circ}{n}$$

$$= r^2 \times n \tan \frac{180^\circ}{n}$$

$$= R^2 \times \frac{n}{2} \sin \frac{360^\circ}{n}.$$

II. Solid Mensuration.

1. Rectangular Solids and Cubes. [Chapter XIII. Page 116.]

(i) *Surface of rectangular solid* $= 2(ab + bc + ca)$.

(ii) *Surface of cube* $= 6a^2$.

(iii) *Volume of rectangular solid* $=$ *length* $\times$ *breadth* $\times$ *height*.

(iv) *Volume of cube* $= (edge)^3$.

2. Prisms. [Chapter XIV. Page 128.]

(i) *Lateral surface of right prism* $=$ (*perimeter of base*) $\times$ *height*.

(ii) *Volume of prism* $=$ (*area of base*) $\times$ *height*.

3. Right Circular Cylinders. [Chapter XV. Page 135.]

(i) *Curved surface of cylinder* $= 2\pi rh$.

(ii) *Volume of cylinder* $= \pi r^2 h$.

4. Pyramids. [Chapter XVI. Page 144.]

(i) *Slant surface of right pyramid*

$$= \frac{1}{2}(\textit{perimeter of base}) \times (\textit{slant height}).$$

(ii) *Volume of pyramid* $= \frac{1}{3}$ (*area of base*) $\times$ *height*.

5. Right Circular Cones. [Chapter XVII. Page 151.]

(i) *Curved surface of cone* $= \pi rl$

$$= \pi r\sqrt{r^2 + h^2}.$$

(ii) *Volume of cone* $= \frac{1}{3}\pi r^2 h$.

6. Frusta of Pyramids and Cones. [Chapter XVIII. Page 159.]

(i) *Slant surface of frustum of right pyramid*

$$= \frac{1}{2}(\textit{sum of perimeters of ends}) \times (\textit{slant thickness}).$$

(ii) *Volume of frustum of pyramid*

$$= \frac{k}{3}\left[E_1 + \sqrt{E_1 E_2} + E_2\right].$$

(iii) *Curved surface of frustum of cone* $= \pi(r_1 + r_2)\,l$.

(iv) *Volume of frustum* $= \frac{\pi}{3}k\left[r_1^2 + r_1 r_2 + r_2^2\right]$.

7. Wedges. Prismoids. [Chapter XIX. Page 168.]

(i) *Volume of wedge* $= \frac{hb}{6}(2a + e)$.

(ii) *Volume of prismoid* $= \frac{k}{6}\left[E_1 + E_2 + 4M\right]$.

8. Spheres. [Chapter XX. Page 172.]

(i) *Surface of sphere* $= 4\pi r^2$.

(ii) *Volume of sphere* $= \frac{4}{3}\pi r^3$.

9. Zones and Segments of Spheres. [Chapter XXI. Page 182.]

(i) *Curved surface of zone or segment* $= 2\pi r h$.

(ii) *Volume of zone* $= \frac{\pi k}{6}\{3(r_1^2 + r_2^2) + k^2\}$.

(iii) *Volume of segment* $= \frac{\pi h}{6}\{3r_1^2 + h^2\}$.

EXAMPLES. XXIV.

EXAMINATION QUESTIONS.

SECTION I.

A. *Questions set in Oxford Local Examinations.*

B. *Questions set to Engineer Students, Militia Officers, Dockyard Apprentices.*

C. *Questions set to Students in Training Colleges.*

D. *Questions set by the College of Preceptors.*

SECTION II.

A. *Questions set for Admission to R.M.C. Sandhurst.*

B. *Questions set for Admission to R.M.A. Woolwich.*

SECTION I.

[A. *Questions set in Oxford Local Examinations.*]

1. Find the area of a quadrilateral ABCD in which the sides AB, BC, CD, DA, and the diagonal AC are respectively 25, 60, 52, 39, and 65.

2. Three equal circles of radius 10 feet touch one another externally, find the area of the space enclosed between them, in square feet correct to two places of decimals. [$\pi = 3\frac{1}{7}$.]

3. The height of a cone is 30 feet. A small cone is cut off the top by a plane parallel to the base: if its volume be $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made? [$\pi = 3\frac{1}{7}$.]

4. A right pyramid stands on a regular hexagonal base, whose side is $20\sqrt{3}$ feet. The total area of the triangular faces is $3000\sqrt{3}$ square feet. What is the height?

5. Two tangents AB, AC to a circle of radius 10 feet contain an angle 60°. Find to the nearest square foot the area included between them and the circle. [$\pi = 3\frac{1}{7}$.]

6. A moat 6 feet deep and 18 feet wide surrounds a circular islet 115 feet in diameter. Find the quantity of water in the moat, taking 1 cubic foot as measuring $6\frac{1}{4}$ gallons. $[\pi = 3\frac{1}{7}.]$

7. Find the volume and the area of the total surface of a truncated circular cone 42 feet in diameter at the base, 21 feet in diameter at the top, and 14 feet high. $[\pi = 3\frac{1}{7}.]$

8. A plank 15 feet long rests vertically against a perpendicular wall. How far must the bottom end be pulled out to lower the top end 3 feet?

9. How often will a wheel 3 feet 4·32 inches in diameter turn round in 2 miles? $[\pi = 3\frac{1}{7}.]$

10. A pendulum swings through an angle of 30° and the end describes an arc of $13\frac{2}{21}$ inches. Find the length of the pendulum. $[\pi = 3\frac{1}{7}.]$

11. From a circular metal disc of uniform thickness with radius of 14 feet a concentric disc is cut of weight equal to the ring remaining. Find the radius of the inner disc. $[\pi = 3\frac{1}{7}.]$

12. A right pyramid 12 feet high, standing on a square base each side of which is 10 feet long, is divided into four parts of equal thickness by planes parallel to the base. Determine the volume of each part in cubic feet.

13. Two spheres of lead, each 10 feet in diameter, are melted down and recast into a right circular cone whose height is equal to the radius of its base. Find its height. $[\pi = 3\frac{1}{7}.]$

14. Plan out and find the acreage of a field from the following

	Links.	
	To E	
	1700	
	1500	400 to D
	900	800 to C
600 to B	600	
	From A	

[B. *Questions set to Engineer Students, Officers of Militia, Dockyard Apprentices, &c.*]

15. The cost of floor-cloth, sufficient to cover a room 15 feet long, was £2. 5*s.*; the value of the floor-cloth being 2*s.* 3*d.* per square yard. How wide was the room?

16. A room is 21 feet long, 13 ft. 6 in. wide, and 10 ft. 6 in. high, and has a doorway 7 ft. $7\frac{1}{2}$ in. by 4 feet, three windows 10 feet by 4 feet each, and a fire-place 4 feet square. Find the cost of painting the walls at 6*d.* per square yard.

17. If $6\frac{1}{4}$ gallons of water equal 1 cubic foot, and water while freezing increases by expansion ·089 of its bulk; how many gallons of water are required to cover a rink 110 yards long and 96 yards wide with ice $5\frac{1}{2}$ inches thick?

18. A cistern 5 feet square contains water 11·09 inches deep. If a gallon $=277\frac{1}{4}$ cubic inches, how many gallons of water does the cistern contain?

19. A section of a stream is 10 feet wide and 10 in. deep; the mean flow of the water through the section is 3 miles an hour; taking 25 gallons to equal 4 cubic feet, find how many gallons of water flow through the section in 24 hours.

20. The diagonals of a rhombus being 88 and 234 feet, find the area; also find the length of a side and the height of the rhombus.

21. Find the volume of a pyramid, the height of which is 12 inches, and the base an equilateral triangle, each side of which is 10 inches.

22. A spherical cannon-ball, 15 inches in diameter, is melted and cast into a conical mould, the base of which is 20 inches in diameter; find the height of the cone.

23. One side of a triangular field is 369 links in length, and the perpendicular upon it from the opposite angle measures 582 links. If the field is let for £6, find to the nearest penny the rent per acre.

24. On the same side of a line 20 feet long, a semicircle and a quadrant are drawn. Find the area included between them. [$\pi=3{\cdot}1416$.]

25. Find to the nearest penny the cost of gilding at $6d.$ per square foot, the slant surface of a cone whose altitude is 3 feet, and the diameter of whose circular base is 3 feet. [$\pi=3\cdot1416$.]

26. In a trapezium, two sides of which are parallel and the other two sides equal, the lengths of the parallel sides are 85·5 feet and 9·5 feet respectively and the equal sides are each 47·5 feet: find the area of the trapezium, in square yards, feet and inches.

27. ABC is an equilateral triangle; two circles are described, each passing through A, one of them touching BC at its middle point, and the other passing through B and C; and each side of the triangle is 14 inches long: find the areas of the circles.

28. A cylindrical cup of external diameter $4\frac{1}{2}$ inches, height $6\frac{1}{4}$ inches, and thickness $\frac{1}{4}$ inch, weighs 24 oz. 9 dwts.: what would be the weight of a hemispherical lid, of the same material and thickness, to fit the top?

29. The perimeter of one square field is 588 yards, and of another 672 yards. Find the perimeter of a third which is equal in area to the other two.

30. A right cylinder of wood, 10 feet long and 2 feet in diameter, is sawn along two planes parallel to each other, and parallel to the axis of the cylinder, each plane being at a distance from the axis equal to one-half of the radius of the cylinder. Compare the volumes of the logs so formed.

31. A conical hole is bored in a sphere of 5 inches radius, so that the vertex of the cone is at the centre of the sphere, and that the portion of the spherical surface removed is one quarter of the whole surface of the sphere. Find the area of the surface of the conical hole.

32. Find the solid and superficial contents of a cylindrical ring whose thickness is 9 inches, and inner diameter 32 inches. [$\pi=\frac{22}{7}$.]

33. Explain the use of Gunter's chain. Find the area in acres of a quadrilateral field of which the diagonal is 1274 links, and the perpendiculars upon it from the opposite angles 550 and 583 links.

34. Find the area of a segment of a circle of which the arc is one-third of the circumference, the radius being $7\frac{1}{2}$ inches. [Take $\pi = 3\frac{7}{50}$, $\sqrt{3} = 1\frac{11}{15}$.]

35. A cube of wood, measuring 2 feet each way, has a square hole cut through it, perpendicular to the top and bottom, leaving each side 3 inches thick; a piece is taken off the top by a saw cut, passing through one of the edges and bisecting the face of the opposite side; the remaining portion weighs 141 lbs. 12 oz. What is the weight of a cubic inch of the wood?

36. The base of a quadrilateral figure is 13·5 feet in length, and is divided into three equal parts by the perpendiculars upon it from the extremities of the opposite side; each of these perpendiculars is 2·5 feet in length; find the area of the quadrilateral figure.

37. Find the volume of the solid which would be generated by the revolution, about its base, of the figure mentioned in the last question. [$\pi = 3\frac{1}{7}$.]

[C. *Questions set to Students in Training Colleges.*]

(*First Year.*)

38. How many panes of glass, each $4\frac{3}{5}$ by $3\frac{3}{5}$ inches, will be required for a window with a semicircular head, whose extreme height is 35 feet and width 14 feet? [$\pi = \frac{22}{7}$.]

39. A rectangular field contains $1\frac{1}{2}$ acres; the distance between opposite corners is $6\frac{1}{2}$ chains; find the length and breadth of the field.

40. If the sides of a right-angled triangle are in Arithmetical Progression, to what numbers are they proportional?

41. Find in yards, to four places of decimals, the difference between the area of a regular hexagon, each of whose sides is 72 yards, and the area of the circle inscribed in it. [$\pi = 3\frac{1}{7}$.]

42. Find, to three places of decimals, the radius of the circle whose area is the sum of the areas of the two triangles whose sides are 35, 53, 66, and 33, 56, 65. [$\pi = 3\frac{1}{7}$.]

43. Draw a plan and find the area of a level field with five straight hedges, from the following notes. (All lengths in links.)

	To E	
To D 120	220	
	310	200 to C
	420	
To B 180	200	
	From A	

44. In a circular riding school of 118 feet in diameter, a circular ride 10 feet wide is to be made just within the outer edge of the building. Find the cost of doing this at $3\frac{1}{2}d.$ per square yard. [$\pi = 3\frac{1}{7}$.]

45. The area of a semicircle is 13013 square feet; find its total perimeter. [$\pi = 3\frac{1}{7}$.]

(Second Year.)

46. A cone and hemisphere have equal bases and equal volumes; find the ratio of their heights.

47. One quarter of the volume of a cylindrical boiler 12 feet long and 5 feet in diameter internally is fitted with cylindrical heating tubes fitted longitudinally and $7\frac{1}{2}$ inches in diameter. Find the number of tubes.

48. A prismoidal tub has a rectangular base 2 feet by 1 foot, and a top 2 ft. 9 in. by 1 ft. 9 in., and is 18 inches deep Find how much water it contains, having given that a cubic foot of water weighs 1000 oz. and a gallon weighs 10 lbs.

49. A minute of latitude contains 6,080 feet and a metre is the ten-millionth part of a quadrant of the meridian. A kilogramme is the weight of the one-thousandth part of a cubic metre of water, and a cubic foot of water weighs 1,000 oz. Express a kilogramme in ounces to two places of decimals.

50. Find the curved surface of the frustum of a cone whose top and bottom diameters are 4 and 6 feet, and slant side 8 feet. [$\pi = 3\frac{1}{7}$.]

51. A sphere, whose radius is $2\frac{1}{4}$ inches, is enclosed in a hollow cylinder of the same radius, whose length is equal to its

circumference; how many cubic inches are there in the remaining part of the cylinder? [$\pi = 3\frac{1}{7}$.]

52. A circular ring fits closely round a cylinder of the same diameter ($2d$); find the height of the cylinder (i) if their volumes are equal, (ii) if the total surface of the cylinder = surface of the ring.

53. Assuming that the volume of a cone is one-third of the base × height, find an expression for the difference between the volumes of two *similar* cones, of which the height and radii of the base are H, R, and h, r respectively. Hence deduce the formula for the volume of a frustum of a cone.

(*Third Year.*)

54. The sides of the base of a triangular prism are 17, 25, and 28 feet, and its height is 20 feet; find its volume.

55. A basin is in the form of a segment of a sphere; if the diameter of the top is $5\frac{1}{4}$ inches, and the depth $1\frac{3}{8}$ inches, find the number of cubic inches it contains.

56. Find the volume of a frustum of a square pyramid, each side of one of its ends being 6 feet, each side of the other end 4 feet, and the perpendicular height 5 feet.

57. Find approximately the diameter of a sphere whose volume is 5948 cubic inches.

58. A right-angled triangle, whose sides are 4 and 3 inches, is made to turn round its hypotenuse; find the volume and surface of the double cone so formed.

59. Find the area of the convex surface of the segment of a sphere, the height being 8 inches, and the diameter of the sphere $10\frac{1}{2}$ inches.

60. A crystal consists of a cube, capped at two opposite faces by pyramids, the sides of which are equilateral triangles. Shew that the whole surface is equal to that of a square on the sum of the diameters of a rhombus of which one of the equilateral triangles is a half.

[D. *Questions set in Certificate Examinations by the College of Preceptors.*]

61. A cricket ground contains 3 acres 1 rood 4 poles 25 yards, and is to be enclosed by a wall. The ground is a square. The cost of the wall is 7*s.* $10\frac{1}{2}$*d.* per lineal yard. What will be the expense?

62. ABCD is a field in the form of a trapezium whose diagonal AC is 320 yards; AD = 144, DC = 360, CB = 240, AB = 180 yards. Draw a fair plan of the field, and then find its area.

63. How many slabs, each measuring 9 feet long by 8 inches broad, will be required to pave a court 120 feet by 96 feet, after deducting for six circular beds, the diameter of each being 8 feet?

64. Describe (if possible by a Field-Book) the measuring of a field ABCDE, F, H, K being the feet of perpendiculars drawn from B and E to AC, and from D to EC. Then find the area in acres, roods, and poles.

[Given: AC = 660 yards, CE = 550 yards, BF = 121 yards, EH = 363 yards, KD = 165 yards.]

65. The content of a cistern in the form of a cube is 15 yards 16 feet 1512 inches. Find the cost of lining its sides and bottom with lead costing 9*d.* per superficial foot.

66. The perpendicular height of a square chimney is 150 feet 3 inches. The side of the base measures 12 feet 6 inches, and the side at top measures 6 feet 3 inches. The cavity is a square prism whose side measures 3 feet 9 inches. How many cubic feet and inches of solid brickwork are there in the building?

67. Take the Earth's diameter = 7958 miles, find its surface and solid content.

68. From the following dimensions &c., draw out a plan of the irregular field. Find the area in acres, roods, and poles. (The measurements are in links.)

Field Book.

	To *G*	
	2560	
To *F* 960	1880	
	1440	160 to *E*
To *D* 700	1200	
To *C* 250	850	
	600	912 to *B*
	From *A*	

69. Find the area of a triangle whose sides are 1840, 1336, and 1520 links, giving the answer in acres, roods, and poles.

70. A field in the form of a trapezoid has its parallel sides equal to 5 chains 15 links and 3 chains 85 links, respectively, and the perpendicular distance between them is 15 chains. Find the area; also draw a figure.

71. Draw plan and find the area of a field from following Field Book:—

	To *B*	
	1000	
To *F* 88	625	
	500	To *E* 66
To *D* 176	375	
	250	To *C* 121
	From *A*	

72. Find area and draw plan of a field from following Field Book:—

	To *B*	
	1200	To *H* 160
To *G* 280	1120	
To *F* 300	960	
	940	To *E* 400
To *D* 760	760	
	200	To *C* 30
	From *A*	

73. The outer diameter of a water-pipe is 2 feet, the inner diameter is 1 foot 8 inches, and the length of the pipe is 40 feet. Find the number of solid inches of metal used in the construction of the pipe.

74. The solid content of a triangular prism is 4·16 cubic feet, and the sides of the triangular end are 2 feet 1 inch, 1 foot 8 inches, and 1 foot 3 inches. Find the length of the prism.

75. Find the solid content of a pyramid whose vertical height is $14\frac{2}{3}$ feet, and which has for its base a triangle whose sides measure 72, 58, and 50 inches.

76. How many cubic feet are there in a wedge whose altitude is 14 inches, the length of its edge 21 inches, the length of its base 32 inches, and the breadth of the base $4\frac{1}{2}$ inches?

77. What is the solidity of a prismoid, the length and breadth of whose greater end are 16 feet and 8 feet; the length and breadth of the less end are 12 feet and 6 feet, and the perpendicular distance is 18 feet?

78. How many cubic feet are there in a pillar whose ends are rectangles, the greater end measuring 64 by 40 inches, and the less end 32 by 8, and the perpendicular height 50 feet?

79. Required the solid content of a cylindrical ring, whose thickness is 27 inches and inner diameter 96 inches.

80. Find the area of an irregular field, and draw a plan fairly to scale, from the following Field Book:—

	AB	
0	2628	31
58	2010	
	1960	13
	1570	31
78	1400	
	1110	38
54	920	
	670	25
84	520	
90	000	58

From *A* go East

SECTION II.

[A. *Questions set for Admission to R. M. C. Sandhurst.*]

Note. *Unless otherwise stated, take* $\pi = 3{\cdot}1416$.

1. Find the area of the triangular field **ABC** from the following measurements on the ordnance survey of 25 inches to the mile.

AC 4·1 inches, perpendicular from **B** on **AC** 1·59 inches. Calculate the same area from the three sides, **AB** measuring 3·3 inches and **BC** 2 inches. Express the mean of the two in acres.

2. A well 5 feet in diameter and 30 feet deep is to have a lining of bricks fitting close together without mortar, 9 inches thick. Required approximately in tons the weight of the bricks, supposing a brick $9 \times 4\frac{1}{2} \times 3$ inches to weigh 5 lbs.

3. A hollow shell 12 inches in diameter is placed in a conical vessel whose vertical angle is 60°, and water poured into it until it just covers the shell and fills the cavity in it. When the shell emptied of the water in it is removed, and a solid ball of the same diameter substituted for it, the water stands $\frac{1}{2}$ inch above it; find approximately the thickness of the shell.

4. A ball of lead, 4 inches in diameter, is covered with gold. Find the thickness of the gold, in order that (1) the volumes of gold and lead may be equal, (2) the surface of the gold may be twice that of the lead.

5. If a right cone on a circular base be divided into three portions by two sections parallel to the base at equal distances from the base and vertex, compare the three volumes into which it is divided.

6. The area of a rectangular field is $\frac{2}{5}$ of an acre; and its length is double its breadth; determine the lengths of its sides approximately.

If a pony is tethered to the middle point of one of the longer sides, find the length of the tether in yards, correct to two places of decimals, in order that he may graze over half the field.

7. An obelisk $68\frac{1}{4}$ feet high has a square section throughout; it is $7\frac{1}{2}$ feet wide at the base, and gradually tapers to a width of 5 feet, the summit being in the form of a pyramid $7\frac{1}{2}$ feet high; it is made of granite, of which a cubic foot weighs 156 lbs.; find the weight of the obelisk.

8. A solid cone 15 inches high is placed on its base in a cylindrical vessel the inner diameter of which is the same as that of the base of the cone; water is then poured into the cylinder until it is 9 inches deep; the cone being removed, another cone, with the same sized base, but of different altitude, is substituted for it; and the surface of the water just reaches the top of this cone. Find the height of the second cone.

9. Sketch plan and calculate the area of a field ABEGFDC from the following notes:

	Yards	
	To G	
	204	
To F 94	198	
	122	10 to E
To D 64	117	
To C 44	88	
	63	70 to B
	From A	

10. A frustum of a pyramid has rectangular ends, the sides of the base being 25 and 36 feet. If the height of the frustum be 60 feet, and its volume 50 180 cubic feet; find the area of the top.

Find, to the nearest foot, the radius of the sphere whose volume is equal to the volume of the frustum. [$\pi = 3\frac{1}{7}$.]

11. Find the cost of the canvas, 2 feet wide, at 2s. 6*d.* a yard, required to make a conical tent, 12 feet in diameter and 8 feet high. [$\pi = 3\frac{1}{7}$.]

12. A public garden occupies two acres, and is in the form of a square. If a pathway goes completely round its inner edge, and occupies one-eighth of an acre, what is its width?

13. A cubical box, 5 feet deep, is filled with layers of spherical balls, whose diameters, where they touch, are in vertical and horizontal lines.

Find what portion of the space in the box would be left vacant if the diameter of a ball is half an inch.

14. A cathedral has two spires and a dome; each of the former consists, in the upper part, of a pyramid 60 feet high, standing on a square base, of which a side is 20 feet. The dome is a hemisphere of 40 feet radius.

Find the cost of covering the three with lead at $7\frac{1}{2}$*d.* per square foot. [Take $\pi = \frac{22}{7}$.]

15. A half-penny piece is one inch in diameter. Six half-pennies are placed so that each coin touches two others, their centres being all on the circumference of a circle. Find the area which they enclose.

16. A circular disk of lead, 3 inches in thickness and 12 inches diameter, is wholly converted into shot of the same density, and of ·05 inch radius each. How many shot does it make?

17. The interior of a building is in the form of a cylinder of 15 feet radius and 12 feet altitude, surmounted by a cone whose vertical angle is a right angle. Find the area of surface and the cubical content of the building.

18. The three conterminous edges of a rectangular block are $9\frac{2}{6}$, $13\frac{1}{5}$, and $14\frac{3}{10}$ inches; find the length of its diagonal.

19. If a cubic foot of cast iron weigh 450 lbs., what will be the weight of a cast iron spherical shell whose external diameter is 6 inches and thickness $\frac{1}{2}$ an inch? [$\pi = \frac{22}{7}$.]

20. The area of an equilateral triangle is 17320·5 square feet. About each angular point, as centre, a circle is described with radius equal to half the length of a side of the triangle. Find the area of the space included between the three circles. [$\sqrt{3} = 1·73205$.]

21. A hollow cone, the length of whose slant side is twice the radius of its base, is held with its vertex vertically downwards and completely filled with water. A sphere of greater density than water is gradually immersed, and it is found that, when it rests upon the sides of the interior of the cone, it is just submerged. Find the amount of water displaced by the sphere, and also the amount contained between the sphere and the vertex of the cone. Consider radius of base of cone as 1·73205 inches.

22. Find the perimeter and the radius of a circle the area of which is 5·309304 square feet.

23. A solid sphere fits closely into the inside of a closed cylindrical box, the height of which is equal to the diameter of the cylinder. Having given the radius of the sphere, write down the expressions for the volume of the sphere, the surface of the sphere, and the volume of the empty space between the sphere and the cylinder.

If the volume of this empty space is 134·0416 cubic inches, what is the radius of the sphere?

24. A right prism on a triangular base—each of whose sides is 21 inches—is such that a sphere, described within it, touches its five faces: find the volume of the sphere, and of the space between it and the surface of the prism. [$\pi = \frac{22}{7}$ and $\sqrt{3} = 1·732$.]

25. Find the expense of paving a circular court 80 feet in diameter, at 3*s.* 4*d.* per square foot, leaving in the centre a space for a fountain in the shape of a hexagon, each side of which is a yard.

26. A hollow right prism stands upon a base which is an equilateral triangle. The vertical faces of the prism are squares— the side of a square being one foot. The prism is filled with water, and the largest possible sphere is then submerged in it. Find (to the nearest cubic inch) the amount of water remaining in the prism. $[\pi = \frac{22}{7},\ \sqrt{3} = 1{\cdot}73.]$

27. A circular disk of cardboard one foot in diameter is divided into six equal sectors by pencil lines drawn through the centre.

In each sector there is described a circle touching the two bounding radii of the sector and also the arc joining their ends at its middle point. If the circles are cut out from the six sectors, find the area of cardboard remaining.

28. A hollow paper cone, whose vertical angle is 60°, is held with its vertex downwards, and in it there is placed a sphere of radius two inches. The portion of the cone remote from the apex is now cut away along the line where the paper touches the sphere. Find the exterior surface of the body thus formed. $[\pi = \frac{22}{7}.]$

29. Prove that the area of a trapezoid is one-half the product of the sum of the two parallel sides by the perpendicular distance between them.

The area of a trapezoidal field is $4\frac{1}{2}$ acres; the perpendicular distance between the parallel sides is 120 yards; and one of the parallel sides is 10 chains; find the other.

30. Express the volume of a cone in terms of the radius of the base and the vertical height.

If the diameters of the circular ends of a frustum of a cone be 4 inches and 6 inches, and the volume of the frustum be 209 cubic inches; find the height of the cone. $[\pi = 3\frac{1}{7}.]$

31. In the pentagonal field ABCDE, the length of AC is 50 yards and the perpendiculars from B, D, and E upon AC are 10, 20, and 15 yards, the distances from A of the feet of the perpendiculars from D and E being 40 and 10 yards; find the area.

32. A sphere of one foot radius rests on a table, find the volume of the right hollow cone which can just cover it, the section of the cone through the axis being an equilateral triangle.

33. 105 half-penny pieces lying on a flat surface with their edges in contact are just contained by a frame in the form of an equilateral triangle. The diameter of a half-penny being one inch, show that the side of the triangle is $(13 + \sqrt{3})$ inches, and calculate its area approximately.

34. Gold is 19·25 times as heavy as water, and a cubic foot of water weighs 997 oz. avd. Find (approximately) how many square feet a cubic inch of gold will cover in the form of gold leaf, given that one grain of gold will cover 56 square inches.

35. Find the area of a triangle, whose sides are 13·6 inches, 15 inches, and 15·4 inches.

Also find (correct to the thousandth part of an inch) the length of one of the equal sides of an isosceles triangle, on a base of 14 inches, having the same area.

36. A circular room, surmounted by a hemispherical vaulted roof, contains 5236 cubic feet of air, and the internal diameter of the building is equal to the height of the crown of the vault above the floor. Find the height.

37. Two pipes, one of lead and the other of tin, are respectively 49 and 61·6 inches long; they both have the same internal diameter, 1 inch; and the external diameter of the lead pipe is 1·2 inch. If lead is 11 times and tin 7 times as heavy as water, what must be the external diameter of the tin pipe that both pipes may have the same weight?

38. Assuming a drop of water to be spherical, and one-tenth of an inch in diameter, to what depth will 500 drops fill a conical wine-glass, the cone of which has a height equal to the diameter of its rim?

39. The minute hand of a clock is 10 inches long. Find the area on the clock face which it describes between 9 a.m. and 9.35 a.m.

40. From a cubic foot of lead is cut out a pyramid whose base is one face of the cube, and whose vertex lies in the face opposite the base. If the remainder of the lead is melted and cast into a sphere, find its diameter.

41. A hill in the shape of a right cone stands on a horizontal plane. At a certain point in the plane the circular base of the cone subtends a right angle, and the elevation of the summit is half a right angle. Show that the slant side of the hill, as seen against the sky, subtends 60° at the same point.

42. Water is flowing steadily through a pipe which consists of two parts. The cross-section of the first part is an equilateral triangle of side 6 inches, and that of the second a circle of radius 2 inches. Find (approximately) the rate of flow of water in the second part, that in the first being 37·7552 feet per minute. [$\pi = \frac{22}{7}$; $\sqrt{3} = 1{\cdot}732$.]

43. Calculate the diameter of a circle equal in area to the curved surface of a right cone, whose vertical angle is 90° and vertical height 1·4142 feet.

44. A solid iron cube, the edge of which is two feet in length, and a solid iron sphere, the radius of which is one foot, are thrown into a cubical tank, which is six feet across and is half filled with water. Find the rise of the surface of the water in inches, to five places of decimals, it being taken for granted that the cube and the sphere are both completely submerged.

45. Find, in feet, to three places of decimals, the radius of a circle, the area of which is equal to the area of a regular hexagon, the side of which is two feet.

Also find the radius of a sphere, the volume of which is equal to the volume of a solid circular cylinder, of height one foot, and radius nine inches.

46. Write down the formulæ for the volume of a cone of height a and base A, and for the volume of a sphere of radius r.

Find to the nearest gramme the weight of a steel body consisting of a cone and a hemisphere with a common base, the height of the cone being equal to the diameter of the base, that is 10 centimetres. [Given steel is 7·8 times as heavy as water, and one cubic centimetre of water weighs one gramme.]

47. A frustum of a regular pyramid has square ends: the edge of the lower end is 10 inches, and that of the upper end 5 inches; and the height of the frustum is $7\frac{1}{2}$ inches. Find the length of a slant edge of the frustum, and the area of the slant faces.

48. (i) Find the number of spherical bullets, each one centimetre in diameter, that can be cast from a regular tetrahedron of lead, an edge of which measures 10 centimetres.

(ii) A circle is described *about*, and a second is inscribed *within* a regular hexagon the length of whose side is 1 foot. Find the area lying between the two circles.

49. Write down the expressions for the volume of a right circular cylinder of radius r and height h, and for the volume of a spherical shell, the external and internal radii of which are 5 inches and 4 inches respectively.

If this shell is made of lead, and if it is filled with water, find to the nearest ounce, its total weight, it being given that a cubic foot of water weighs 1000 oz. and that lead is 11·5 times as heavy as water.

50. The height of a solid right cone is 14 inches, and its vertical angle is two-thirds of a right angle. If this cone is cut into two parts by a plane bisecting the axis at right angles, find to four places of decimals the volume and total slant surface of each part.

[B. *Questions set for Admission to R.M.A. Woolwich.*]

NOTE. *Most of the following questions require the use of Mathematical Tables. In solving them, Chambers' Tables have been employed; and the results are given correct to six significant digits.*

Unless otherwise stated, take $\pi = 3{\cdot}1416$, $\log \pi = {\cdot}4971499$.

51. Determine the diameter of a cylindrical gasholder to contain 10 million cubic feet of gas, supposing the height to be made equal to the diameter; and determine in tons the weight of iron plate, weighing $2\frac{1}{2}$ pounds per square foot, required in the construction of the gasholder, supposing it open at the bottom, and closed by a flat top.

52. Determine the number of cubic yards in a bank of earth on a horizontal rectangular base 60 feet long and 20 feet broad, the four sides of the bank sloping up to a ridge at an angle of 40° to the horizon.

53. It is proposed to add to a square lawn measuring 58 feet on a side two circular ends, the centre of each circle being the point of intersection of the diagonals of the square. How much turf will be required for the purpose?

54. A hollow pontoon has a cylindrical body 20 feet long, and hemispherical ends, and is made of metal $\frac{1}{8}$ of an inch thick. The outside diameter is 3 ft. 4 in. Find its weight, having given that a cubic inch of the metal weighs 4·5 oz.

55. Taking $\pi = 3{\cdot}14159$, calculate approximately the area and the perimeter of the circle inscribed in a square, the side of which is 359·5678 feet.

56. A vessel in the shape of a circular cylinder, open at the top, the height of which is double its diameter, stands on a horizontal table, and is filled with water. A solid circular cone, of the same base and the same height, is pushed into the water until its vertex reaches the base of the cylinder, and is then taken out. Find the height at which the water afterwards stands in the cylinder.

57. State formulæ for the areas of the curved surfaces of a sphere, of a right circular cylinder and of a right circular cone.

How many square yards of canvas are required to make a conical tent 9 feet high, such that a man of 6 feet could stand anywhere inside, within a radius of 2 feet from the centre without stooping?

58. A pint tankard is in the form of a frustum of a circular cone; its height is $4\frac{1}{2}$ inches and the diameter of its base $3\frac{1}{2}$ inches, both measurements taken inside. Find the diameter of the top, being given that a gallon of water weighs 10 lbs. and a cubic foot of water weighs 1,000 oz. Employ logarithm tables to find the dimensions of the similar quart tankard to four places of decimals.

59. Prove that the area of a sector of a circle is equal to half the product of the length of the arc and the radius.

If the area is 2240·567 square feet, and the radius 33·495 feet, calculate the length of the arc.

60. An iron boiler is constructed in the form of a cylinder with hemispherical ends. If the radius of the boiler is one foot, and its extreme length (inside) four feet, calculate, as a fraction of a ton, the weight of water which it will hold, assuming that 1000 oz. is the weight of a cubic foot of water.

61. A railway tunnel consists of a hollow semi-cylindrical top, terminating below in a trough with slanting sides and flat base. The radius of the former being 12 feet, the base and height of the latter being 20 and 18 feet respectively, and the length of the tunnel 1,200 yards, find the cost of facing the sides and roof with brick at 1*s.* 6*d.* per square foot. $[\pi = \frac{22}{7}.]$

62. Within a hollow sphere of 1 foot radius is placed a right prism, the ends of which are equilateral triangles. The side of one of these being 1 foot in length, and the surface of the sphere being in contact with all the six angular points of the prism, find in cubic inches the volume of the latter.

63. A piece of wood is in the form of a regular pyramid on a square base; the side of the base is 6 inches, and the perpendicular distance of the vertex from the base is 8 inches; find the number of cubic inches in the volume of the wood, and the number of square inches in its surface.

64. A cylindrical boiler is hemispherical at its two ends; its radius is 2 feet, and its total length is 8 feet; assuming that a cubic foot of water weighs 62·5 lbs., find the number of tons of water which will fill the boiler. [Take $\pi = 3{\cdot}14$.]

65. The height of a conical tent is $7\frac{1}{2}$ feet, and it is to enclose 200 square yards of ground; find how much canvas will be required. $[\pi = \frac{22}{7}.]$

66. The silk covering of an umbrella forms a portion of a sphere of $3\frac{1}{2}$ feet radius, the area of the silk being $14\frac{2}{3}$ square feet. Find the area of the ground sheltered from vertical rain when the stick is held upright. $[\pi = \frac{22}{7}.]$

67. Find the area of the surface (including the ends) of a hexagonal prism, whose height is 8 ft., the base being a regular hexagon with a side of length 3 ft.

68. The radii of the internal and external surfaces of a hollow spherical shell of metal are 3 ft. and 5 ft. respectively. If it be melted down and the material formed into a cube, find an approximate value for the length of an edge of the cube.

69. Two cylindrical vessels are filled with water; the radius of one vessel is six inches and its height one foot, and the radius of the other is eight inches and its height one foot and a half; find the radius of a cylindrical vessel eleven inches in height which will just contain the water in the two vessels.

70. Having given that the length of each edge of a regular tetrahedron is four inches, determine to three places of decimals, the number of square inches in the total surface of the tetrahedron.

Also find the number of cubic inches in the volume of the tetrahedron.

71. A cylindrical boiler, terminated by plane ends, is internally 15 feet long and 4 feet in diameter, and is traversed lengthwise by 50 cylindrical fire tubes, each 3 inches in external diameter; determine the volume of water the cylinder could contain, taking π to be $\frac{22}{7}$.

72. Supposing an ice field to exist round one of the earth's poles, extending 5° from the pole in all directions, find the area of the ice field in square miles, taking the earth's radius to be 4,000 miles, cos 5° to be ·996195 and $\pi = \frac{22}{7}$.

73. Find the area enclosed by 200 hurdles placed so as to form a regular polygon of 200 sides, the length of each hurdle being 6 feet.

74. A leaden sphere one inch in diameter is beaten out into a circular sheet of uniform thickness $= \frac{1}{100}$th inch. Find the radius of the sheet.

75. Find the area of the greatest circle which can be cut out of a triangular piece of paper whose sides are 3, 4, 5, feet respectively.

76. A conical extinguisher, whose section through the vertex is an isosceles triangle with vertical angle 30°, is placed over a cylindrical candle whose diameter is one inch, and rests so that the point of contact of the top of the candle with each generating line of the cone bisects that line. Find the whole inside surface of the extinguisher.

77. The radii of the circular faces of a frustum of a right cone are 12 and 8 feet, and the area of its curved surface is $20\pi\sqrt{241}$ square feet; find the thickness of the frustum. Show that the vertical angle of the cone, of which this is a frustum, is 29° 51′ 46″.

ANSWERS.

EXAMPLES. I. A.

1. 14·811. 2. 9·869. 3. ·2595. £2. 18s. 1d.
5. £14. 13s. 11d.

EXAMPLES. I. B.

1. 53·42. 2. ·02056. 3. ·0004604. ·31831.
5. ·1336.

EXAMPLES. II. A.

1. (i) 17 ft., (ii) 37 ft., (iii) 4 ft. 2 in., (iv) 9 yds. 2 ft., (v) 5 chains, (vi) 370 links, (vii) 3 p. 5 yds. 0 ft. 6 in., (viii) 5 p. 0 yd. 2 ft. 6 in.
2. (i) 21 ft., (ii) 77 in., (iii) 2 ft. 9 in., (iv) 21 yds. 2 ft., (v) 2 chains 9 links, (vi) 9 chains 10 links, (vii) 4 yds. 2 ft.
3. 60 ft. 4. 193 miles. 5. 60 knots. 6. 25·16 feet.
7. 7 chains 23 links. 8. 17 miles. 9. 20 ft. 10. 950 ft.
11. 15 in.; 20 in. 12. 4 in.; 5½ in. 13. 5 hours.
14. 6 miles an hour.

EXAMPLES. II. B.

1. 160 ft. 2. 70·71 ft. 3. 28 ft. 3 in.
4. 5 chains 74 links. 5. 373 ft. 6. 155 yds. 1 ft. 8 in.
7. 17 miles. 8. 20 miles. 9. 28 ft. 10. 51 ft.
11. 6·6 ft. 12. 62 ft. 13. 36 ft. 14. 15 hours.
15. £1. 19s. 7d. 16. 15 min. 33 sec. 17. 33 ft.; 44 ft.
18. 48 in.; 20 in. 19. 3·46 in. 20. 2·45 in.
21. 12 ft.; 5 ft. 22. 15 ft.; 8 ft.

EXAMPLES. III. A.

1. (i) 44 sq. ft., (ii) 152 sq. ft., (iii) 129·375 sq. ft.

2. (i) 30 acres, (ii) $12\frac{1}{2}$ acres, (iii) $16\frac{1}{2}$ acres, (iv) 10·636 acres, (v) 1·89 acres.

3. (i) 18 sq. ft. 9 sq. in., (ii) 160 acres, (iii) 12·769 acres, (iv) 2 r. 1 p., (v) 47 ac. 1 r. 9 p.

4. 13·225 acres.

5. £150.

6. £3. 1s. $10\frac{1}{2}d.$

7. £4. 13s. 4d.

8. 544 sq. yds. 4 sq. ft.

9. 1 ac. 3 r. 9 p.

10. £156. 5s.

11. 53 ac. 0 r. 35 p. 18 sq. yds.

12. 17 ac. 2 r. 21 p. 22 sq. yds.

13. 121 ac. 1 r. 27 p. 17 sq. yds.

14. 1 r. 11 p. 29 sq. yds.

15. £359. 3s.

16. 44 ac. 2 r. 16 p. 26 sq. yds.; £22. 8s. 4d.

EXAMPLES. III. B.

1. (i) 15 ft., (ii) 18 ft. 6 in., (iii) 40 yds., (iv) 68 yds. 2 ft.

2. (i) 17 chains, (ii) 21 chains 25 links, (iii) 16 chains 6 links.

3. (i) 17 ft., (ii) 12 yds. 1 ft., (iii) 110 yds., (iv) 308 yds., (v) 190 chains, (vi) 81 chains, (vii) 1820 links.

4. 52 ft.

5. 60 chains.

6. 15 min.

7. £114. 19s.

8. 35 ft.

9. 15 chains; 5 chains.

10. 69 chains; 17 chains 25 links.

11. 64 yds.

12. 212 yds. 1 ft.

13. 20 ft. 6 in.

14. 15 chains 25 links.

15. 398 yds.

16. 45 chains; 15 chains.

17. 70 yds. 2 ft. 9 in.

18. 7 chains 44 links.

19. 12 chains 15 links.

20. 19 chains 66 links.

EXAMPLES. III. C.

1. 32 sq. ft.

2. 242 sq. ft.

3. 20 p. 19 sq. yds. 2 sq. ft.

4. 9 ft.; 108 sq. ft.

5. 65 ft.; 4680 sq. ft.

6. £102. 7s. 6d.

7. 13·4 acres.

8. 90 acres.

9. 25 min.

10. £110. 11s.

11. 195 sq. ft.

12. 36 yds.

13. 1728.

14. 4 ft.

15. 11 ft.

16. £3.

17. £36.

18. $269\frac{1}{2}$ ft.

19. $31\frac{1}{2}$ yds.

20. 27 in.

21. £31. 10s.

22. 58 ft.; 42 ft.

23. $231\frac{7}{8}$ sq. ft.

24. 11 ft.; 42 ft.

25. 1 ft. 10 in.; 20 ft. 10 in.

26. 232 sq. yds.

27. 132 sq. yds. 4 sq. ft.

28. 1400 sq. yds.; 444 sq. yds. 4 sq. ft.

29. $36\frac{3}{4}$ yds.; 9s. $5\frac{1}{2}d.$

EXAMPLES. III. D.

1. 10 ac. 0 r. 17 p. 20 sq. yds. 2. 18 ac. 2 r. 17 p.

3. $\frac{a^2b^2}{90}$ acres. 4. $\frac{44b}{c}\sqrt{\frac{a}{10}}$. 5. $\frac{abc}{36d}$ ft.

6. $\frac{abs}{5l}$ in. 7. $\frac{5lc}{sa}$ ft. 8. 15 ft.; 10 ft.

9. 15 chains; 13 chains. 10. 15 ft.; 8 ft.

11. 12 chains; 5 chains. 12. $7\frac{3}{4}$ min. 13. 30 ft.; 10 ft.

14. 50 chains; 40 chains. 15. 5 ft. 16. 1 ft.

17. £2. 11*s*. 18. $12\frac{1}{2}$ sq. ft. 19. 720.

20. 3[illegible]2 sq. ft. 21. 162 sq. ft.

EXAMPLES. IV. A.

1. (i) 20 sq. ft., (ii) 48 sq. yds. 8 sq. ft., (iii) 15 acres.
2. (i) 210 sq. ft., (ii) 9 sq. yds. 3 sq. ft., (iii) 12·54 acres.
3. 110 yds. 4. 8 p. 5. (i) 17 ft.; (ii) 10 chains.
6. (i) 272 sq. ft., (ii) 7 sq. yds. 7 sq. ft., (iii) 1 r. 2 p. $7\frac{1}{2}$ sq. yds., (iv) 5·985 acres.
7. (i) 24 ft.; 240 sq. ft., (ii) 5 chains 60 links; 1·848 acre.
8. (i) 7 ft.; (ii) 84 chains. 9. 2 p.
10. (i) 2·4 ft., (ii) 6 chains 72 links, (iii) 14·5 ft.
11. (i) 126 sq. ft., (ii) 84 sq. ft.
12. (i) 34 sq. yds., (ii) 22 sq. yds. 6 sq. ft., (iii) 9 acres, (iv) 225 acres, (v) ·084 acre.
13. 12 ft. 14. 20 yds. 15. 35 yds.
16. (i) 43·3 sq. ft., (ii) 140·3 sq. ft., (iii) 27·06 acres.
17. 48 sq. ft. 18. 120 sq. ft. 19. 25 ft. 20. 13 ft.
21. (i) £16, (ii) £35. 15*s*., (iii) $101\frac{7}{15}$ yds.
22. 25 chains; 7 chains; 117·84 yds.
23. £127. 10*s*.; 30 chains 60 links. 24. £3. 6*s*. 8*d*.

EXAMPLES. IV. B.

1. ·1848 acre. 2. ·791875 sq. mile.
3. (i) 2 ac. 2 r. 29 p. 10 sq. yds., (ii) 4 ac. 0 r. 38 p. 25 sq. yds.
4. £9. 16*s*. 2*d*. 5. 12 yds. 11 in. 6. 52 ft. 8 in.
7. 48 ft. 1 in. 8. 533 yds. 1 ft. 5 in. 9. 8·314 acres.
10. 75 ft. 11. 13 ft. 2 in. 12. 6·32 in.
13. 235 yds. 14. 28 ft. 10 in. 15. 1492·8 sq. ft.

16. 5 sq. ft. 28 sq. in. 17. 21, 28, 35 ft.; 294 sq. ft.

18. 91, 104, 117 yds. 19. 39 ft., 60 ft.; 63 ft.

20. 65 chains; 60 chains; 25 chains. 21. 10 ft.; 8 ft.; 6 ft.

22. 17 ft.; 15 ft.; 8 ft. 23. 21 ft.; 17 ft.

24 20 ft.; 13 ft. 25. 27·81 sq. ft.

26. 17 chains 50 links. 27. 17 ac. 1 r. 11 p. 8 sq. yds.

EXAMPLES. V. A.

1. (i) 168 sq. ft., (ii) 10½ sq. ft. 2. (i) 21½ sq. ft., (ii) 4 ac. 0 r. 12 p.
3. (i) 17 sq. ft.; (ii) 120 acres. 4. 8 chains 33 links.
5. 86·60 sq. in. 6. 2 ft. 1 in.; 2 sq. ft. 49 sq. in.
7. 40 ft. 8. 85 ft. 9. £120. 2s. 7d. 10. 360 sq. ft.
11. 4 ac. 2 r. 19 p. 6 sq. yds. 12 2 sq. ft. 98·4 sq. in.; 19·32 in.
13. 6 sq. ft. 87·17 sq. in.; 23·78 in.
14. 30°, 150°; 9 in. 15. 60°.

EXAMPLES. V. B.

1. (i) 27 sq. ft., (ii) 37 sq. yds. 2. 40 ac. 2 r. 16 p.
3. 269½ yds. 4. 10 chains. 5. £5302. 10s.
6. £5. 7. 17 chains 50 links. 8. 750 sq. yds.
9. 7 ac. 2 r. 32 p. 10. 779·42 sq. ft.

EXAMPLES. V. C.

1. 180 sq. in. 2. 4¼ acres. 3. 61 ac. 0 r. 32 p.
4. £63. [illegible]s. 9d. 5. 210 sq. in. 6. 29 ac. 1 r. 24 p.
7. 2 sq. yds. 8. 20 ac. 0 r. 24 p. 9. 234 sq. ft.
10. 516 sq. ft. 11. £24. 1s.
12. 2 ac. 0 r. 16 p. 29 sq. yds. 13. 10·31 acres.
14. 1418·53 sq. ft. 15. 110 yds.; 66 yds.
16. 12·02 acres. 17. 30°.

EXAMPLES. VI. A.

1. (i) 44 in., (ii) 22 in., (iii) 11 ft., (iv) 29 yds. 1 ft., (v) 8 yds. 1 ft. 8 in., (vi) 12 p., (vii) 45 chains 76 links, (viii) 6 chains 60 links.
2. (i) 42 in., (ii) 11 yds. 2 ft., (iii) 10 p. 1 yd., (iv) 1 chain 26 links.
3. 55 in. 4. 7 in. 5. £29. 6s. 8d. 6. 7 in.
7. 9 miles 3 furlongs. 8. 10. 9. 2800.

10. 30 miles an hour. 11. 30 miles an hour.
12. (i) 154 sq. in., (ii) $38\frac{1}{2}$ sq. in., (iii) 9 sq. ft. 90 sq. in.,
(iv) 68 sq. yds. 4 sq. ft., (v) 5 sq. yds. 7 sq. ft. 58 sq. in.,
(vi) 11 sq. p. $13\frac{3}{4}$ sq. yds., (vii) 3 ac. 3 r. 16 p.,
(viii) 1 r. 15 p. 13·31 sq. yds.
13. £42. 7*s*. 14. £5467.
15. (i) 7 in., (ii) 35 ft., (iii) 4 yds. 2 ft., (iv) 7 chains.
16. 6 in. 17. 5 chains. 18. 12 yds. 1 ft. 19. 17 in.
20. 15 in. 21. (i) $38\frac{1}{2}$ sq. in., (ii) $962\frac{1}{2}$ sq. ft.
22. (i) 88 in., (ii) 1[illegible] yds. 23. 330 sq. in.
24. 5 sq. ft. 94 sq. in. 25. 281[illegible] sq. yds.
26. 44 sq. in. 27. £2. 9*s*. 7*d*. 28. £1[illegible]. 19*s*. 1*d*.
29. 5 ac. 2 r. 7 p. 1 sq. yd. 30. 804 lbs. 9 oz.

EXAMPLES. VI. B.

1. 10 miles. 2. 70·028 yds. 3. 3·141 . . .
4. $18\frac{1}{2}$ miles per second, nearly. 5. 15 ft., $3\frac{1}{3}$ ft.
6. $96\frac{1}{4}$ acres. 7. £2041. 12*s*. 8. ·364, 78 sq. in.
9. 8·86 in. 10. 1 chain 11 links. 11. 3·141 . . .
12. £3. 13*s*. 4*d*. 13. 7 : 8. 14. 8 in.
15. 1 in. 16. 2 in. 17. 4 ft.
18. 7 ft. 19. 154 sq. in. 20. 12 in.
21. $28\frac{7}{8}$ sq. in. 22. $9\frac{5}{8}$ sq. in. 23. $2\frac{1}{3}$ in.
24. 21. 25. 1 : $\sqrt{2}$; 70·71 in. 26. 9·05 in.

EXAMPLES. VI. D.

1. 50·26 in. ; 71·08 in. 2. 113·09 sq. in. ; 226·19 sq. in.
3. 345·71 yds. ; 3·99 acres. 4. 49·49 in. ; 70 in.
5. 33·8 in. 6. 70 in. 7. 462 sq. in. ; 1448 sq. in.
8. 21 in. ; 42 in. 9. 42 in. 10. 27 : 16, 27 : 32.
11. (i) 3 in. ; $8\frac{1}{2}$ in., (ii) 5 in. ; $18\frac{2}{3}$ in.
12. (i) $4\frac{3}{8}$ in. ; $10\frac{5}{8}$ in., (ii) 5 ft. 8 in. ; 30 ft. 10 in.
13. 25 in., 24 in., 7 in. 14. 13 in., 14 in.

EXAMPLES. VII. A.

1. 12 in. 2. 9 ft. or 145 ft. 3. 10 ft. 10 in.
4. 1·21 in. 5. 25 in. 6. 11 yds. 1 ft.
7. 9 in. 8. 1 ft. 3 in. 9. 14 ft. 7 in.
10. 44·64 in. 11. 17·89 in. 12. 28·28 in.
13. 1·876 in.

EXAMPLES. VII. B.

1. 22 in. 2. 22 in. 3. 33° 45′.
4. 50° 24′. 5. 105 in. 6. 522 yds. 2 ft.
7. 291°. 8. 576 miles. 9. 14°·6.
10. 3·141. 11. 56 in. 12. 50⅔ in.
13. 20 ft. 11 in. 14. 49·17 in. 15. 48·91 in.
16. 20·09 in.; ·0024.

EXAMPLES. VIII. A.

1. (i) 231 sq. ft., (ii) 77 sq. ft. 2. 30 ac. 3 r. 8 p.
3. 1·37 in. 4. (i) 272 sq. in., (ii) 7 sq. yds. 7 sq. ft.
5. 84 in.; 15′. 6. 21 ft. 7. 22 ft.
8. 1 chain 92·5 links. 9. 1·74 ft.
10. 2 ft. 10 in. 11. 10 in. 12. 5 min.
13. 39·27 sq. in. 14. 30270 sq. in.

EXAMPLES. VIII. B.

1. 7 sq. in. 2. 10·01 sq. in. 3. 7·53 sq. in.
4. 9·37 sq. in. 5. 5190 sq. in. 6. 168·16 sq. in.
7. 82·53 sq. in. 8. 17·18 sq. in. 9. 1·568 sq. in.
10. 1·305 sq. in. 11. 15·27 sq. in. 12. ·5708 sq. in.

EXAMPLES. VIII. C.

1. 9·06 sq. in. 2. 13·73 sq. in. 3. 34·76 sq. in.
4. 23 sq. in. 5. 102·73 in. 6. 30·71 sq. in.
7. 4·53 sq. in. 8. 20·82 in. 9. 78·54 sq. in.
10. 50·26 sq. in. 11. 42·06 sq. in. 12. 6·16 sq. in.
13. 8·62 sq. in. 14. 2·93 in.
15. 1·72 in.; 58·28 in.; 9·26 sq. in. 16. ·718 in.; 139·28 in.
17. 3 in.; 28·27 sq. in. 18. (i) 21·54 in.; (ii) 1·54 in.

EXAMPLES. IX.

1. (i) 3 in., (ii) 17 in., (iii) 9 in.,
(iv) 11 in., (v) 12 in.; 13 in., (vi) 7 in.; 37 in.,
(vii) 5 in.; 24 in., (viii) 29·6 in., (ix) 10 ft.
2. 90 ft. 3. 55 ft. 4. 30 ft.; 4 ft.
5. 2 ft. 4 in. 6. 4 in.; 3·2 in. 7. 1·7 in.; 5·4 in.
8. 5·625 in. 9. 9·75; 10·5; 11·25. 10. 20; 23·2; 28·8.
11. 106 ft. 12. 85 ft. 9 in. 13. 1·73 sq. in.

14. 10·39 sq. in. 15. 10 sq. ft. 16. 3 in.
17. 3·75 sq. in. 18. 3·36 sq. in. 19. 14·4 sq. in.
20. 210 acres. 21. 1[illegible]00 sq. yds. 22. 10 acres.
23. 704 yds. 24. £200.
25. $\frac{1}{792}$; 1 mile. 26. 25 in. to the mile.
27. 70·70 in. 28. 57·73 in.; 81·65 in.
29. 47·25 sq. in. 30. 230·4 sq. ft.
31. 128 sq. in.; 1 sq. in. 32. $\frac{1}{4096}$.
33. $50\frac{2}{7}$ sq. in.; the sixth. 34. $254\frac{4}{7}$ sq. in.; 280·59 sq. in.

EXAMPLES. X.

1. 43·01 sq. in. 2. 492·43 sq. in. 3. 98·51 sq. in.
4. £2. 16*s*. 7*d*. 5. £80. 9*s*. 6*d*. 6. 10·00 in.
7. 27·71 in. 8. 3·31 in. 9. 6·21 in.
10. 262·89 sq. in. 11. [illegible] in. 12. 314·16 sq. in.
13. 283·17 sq. ft. 14. $nb\left\{a+b\tan\frac{180^\circ}{n}\right\}$.
15. 131·44 sq. in. 16. 10 in. 17. 10 in.
18. 186·51 sq. in. 19. 1558·845 sq. in.
20. $a\left(1+\cos\frac{180^\circ}{n}\right)$; $na^2\cos^4\frac{90^\circ}{n}\cot\frac{180^\circ}{n}$; $\pi a^2\cos^4\frac{90^\circ}{n}\cot\frac{180^\circ}{n}$.
21. 1039·23 sq. in. 22. $\pi r^2\operatorname{cosec}^2\frac{180^\circ}{n}$.

EXAMPLES. XI. A.

1. 1152 sq. ft. 2. 1 sq. ft. 30 sq. in.
3. 57 sq. yds. 3 sq. ft. 4. 9 acres. 5. 112 sq. yds.
6. 12500 sq. yds. 7. 113 sq. yds.
8. 63 ac. 3 r. 24 p. 9. 2[illegible]4 sq. yds.

EXAMPLES. XI. B.

1. 2 r. 31·7 p. 2. 1 ac. 0 r. 7·4 p. 3. 2 ac. 1 r. 9·6 p.
4. 2 ac. 1 r. 11·9 p. 5. 17 ac. 0 r. 10·2 p.
6. 8 ac. 1 r. 18·8 p. 7. 2 r. 1·9 p.

EXAMPLES. XII.

1. 2·57. 2. ·023. 3. ·005.
4. 1·55. 5. 2·63. 6. 5·96.

EXAMPLES. XIII. A.

1. 1300 sq. ft. 2. 96 sq. ft. 3. 810 sq. ft.
4. $121\frac{1}{2}$ sq. ft. 5. 10*s*. 6. 71 sq. yds. 1 sq. ft.
7. £3. 8. $7\frac{2}{3}$ sq. yds 1 sq. ft. 9. 168 yds.
10. £3. 15*s*. 11. $2\frac{1}{2}$*d*. 12. 5 ft. 3 in.; $165\frac{3}{4}$ sq. ft.
13. £3. 7*s*. 4*d*. 14. 5*s*. 2*d*. 15. 100.
16. (i) 1 ft. 3 in., (ii) 4 ft. 9 in. 17. 1 ft. 1 in.
18. (i) 240 c. ft., (ii) 70 c. ft.
19. (i) 274 c. ft. 1080 c. in. (ii) 52 c. yds. 19 c. ft. 1431 c. in.
20. £105. 21. 400 gallons. 22. 25 tons.
23. 128 days. 24. 8640. 25. 450 gallons.
26. (i) 8 in., (ii) 1 ft. 3 in., (iii) 5 ft. 4 in. 27. 1 ft. 6 in.
28. $4\frac{1}{2}$ in. 29. 1 ft.

EXAMPLES. XIII. B.

1. 241 in. 2. 77 sq. in. 3. $\frac{1}{2}$ in.
4. 5 in. 5. 21 ft., 14 ft. 6. 13 ft. 6 in.
7. 25 in.; 20 in.; 15 in. 8. 10 in. 9. 16 ft.; 15 ft. 10. 17 ft.
11. 12 ft., 11 ft., 10 ft. 12. 15·36 in. 13. 31 in. 10 s.
14. 10 ft. 4 in. 15. 4·01 in. 16. 376 lbs. $13\frac{1}{3}$ oz.
17. 304·8 lbs.; 663·7 lbs 18. 35·78 in. 19. 2 ft. 6 in.
20. 16 sq. ft. 96 sq. in. 21. 3 ft.; 4 ft. 6 in.; 7 ft. 6 in.
22. 66·53 in. 23. 17·60 in.
24. 17·12 in. 25. 4 sq. ft. 99 sq. in.; 1193·2 c. in
26. 34·64 in. 27. $12\frac{1}{2}$ sq. ft., 3 c. ft. 12 c. in. nearly
28. 1 ft. 4 in. 29. 21 in., 16 in. 30. 61 in.

EXAMPLES. XIV.

1. 5 sq. ft. 2. 15 sq. ft. 3. £3.
4. 48 c. in., 108 sq. in. 5. 840 c. in.; 4 sq. ft. 12 sq. in
6. 62·016 ounces. 7. 110 c. in.; 1 sq. ft. 8. ·375 c. ft.
9. 15,000 gallons. 10. 150 tons. 11. 13 ft.
12. 2 ft. 6 in. 13. 3 ft. 14. $11\frac{1}{2}$ sq. ft.; 2 c. ft. 756 c. i
15. 15 in.; 996 sq. in. 16. £1665.
17. 7 in. 18. 900 c. in. 19. 26·27 ft.
20. 21 min. 36 sec. 21. 3·84 in. 22. 9·5 in.
23. 4 c. ft. $245\frac{1}{2}$ c. in.; ·171 ... 24. 15 in.; 8 in.

EXAMPLES. XV. A.

1. (i) 220 sq. in., (ii) 36 sq. ft. 96 sq. in.
2. (i) 253 sq. in., (ii) 23 sq. yds. 0 sq. ft. 68 sq. in.
3. 14 in. 4. £6. 12s. 5. 550 sq. yds.
6. (i) 770 c. in., (ii) 32 c. ft. 144 c. in. 7. 2677[illegible]/8 lbs.
8. 66 gallons. 9. £2722. 10s. 10. (i) 8 in., (ii) 3 ft. 0 in.
11. 31 ft. 6 in. 12. 200 ft. 13. $5\frac{1}{2}$ in.
14. 5·6 ft. 15. 7128 sq. in.; 6600 c. in.
16. 67 lbs. 17. 4752 lbs.

EXAMPLES. XV. B.

1. 1100 c. in. 2. $8\frac{1}{4}$ sq. ft. 3. £16. 10s.
4. 308 sq. in. 5. 1650. 6. ·1 in.
7. $12\frac{1}{4}$ in. 8. ·006 in. 9. 3 min. 34 sec.
10. 2·21 in. 11. 56·5 ft.
12. 736·90 sq. in., 1507·97 c. in. 13. 1885 in. nearly. 75·5 oz.
14. 1609 tons. 15. 146·7 ft., 101·5 lbs. 16. 2 in.
17. ·1 in. nearly. 18. 1 in., or 2 in.

EXAMPLES. XVI. A.

1. 20 sq. ft. 2. 10 sq. ft. 3. 8 sq. ft. 128 sq. in.
4. £3. 3s. 9d. 5. (i) 50 c. in., (ii) 128 c. in., (iii) 90 c. in.
6. (i) 7 in., (ii) 1 ft. 7. 8 in. 8. 1 ft. 6 in.
9. 2 tons 9 cwt. 12 lbs. 10. £34.
11. 1726 lbs. 9 oz. 12. 2436 sq. ft.

EXAMPLES. XVI. B.

1. 223·6 sq. in. 2. 1 ft. 4 in. 3. 46·80 sq. in.
4. 32 sq. in. 5. 11·31 in.; 443·40 sq. in.
6. 15·65 in.; 831·55 sq. in. 7. 2 ft. 3 in. 8. ·729 c. in.
9. ·5 in. 10. 500 c. in. 11. 10392·3 c. in.
12. 1 ft. 8 in. 13. ·2122.
14. $\frac{1}{6}a^3$ c. in.; $\frac{\sqrt{3}}{2}a^2$ sq. in.; $\frac{a}{\sqrt{3}}$ in.
15. $\frac{1}{6}abc$ c. in.; $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$ sq. in.; $\frac{abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ in.

16. $6\frac{2}{3}$ c. ft. 18·93 sq. ft. **17.** $\frac{a^2}{a+b}$ in. **18.** 6 in.

19. 10 in. **20.** 34·95 sq. in. **21.** $\frac{a^3\sqrt{2}}{3}$ c. ft. **22.** 10 ft.

EXAMPLES. XVII. A.

1. (i) 330 sq. in., (ii) 28 sq. ft. 38 sq. in.
(iii) 216 sq. ft. 70 sq. in., (iv) 3 sq. ft. 118 sq. in.

2. 1 ft. 9 in. **3.** 7 in. **4.** £3. 2s. 4d. **5.** 61 sq. yds. 1 sq. ft.

6. (i) 66 c. in., (ii) 616 c. in., (iii) 1232 c. in., (iv) $60\frac{1}{2}$ c. ft. **7.** (i) 1 ft. 9 in., (ii) 2 ft. 4 in.

8. (i) 1 ft., 3 ft. 1 in., (ii) 2 ft. 9 in., 5 ft. 5 in.

9. $42\frac{7}{8}$ gallons. **10.** 149 lbs. **11.** 2 tons 3 cwt. 24 lbs.

12. 3·84 ft. **13.** 17·32 in.

14. 5 c. ft. 1216 c. in.; 15 sq. ft. 40 sq. in. **15.** 329·12 sq. in.

EXAMPLES. XVII. B.

1. 389·56 sq. in. **2.** 204·20 sq. in. **3.** 1016·62 sq. in.

4. 3·90 in. **5.** 27 ft. **6.** 1 ft. 3 in.

7 36·26 gallons. **8.** 16 min. 40 sec. **9.** 32·67 in.

10. $1822\frac{2}{7}$ sq. in. **11.** 1232 c. in. **12.** 44·91 c. in.

13. 1914 sq. in. **14.** 23·63 in. **15.** 42·43 c. in.

16. 24·25 in.; 14 in. **17.** 16·8 in. **18.** 9·5 in.; 19 in.

EXAMPLES. XVIII. A.

1. 110 sq. in. **2.** 6 sq. ft. 126 sq. in.

3. 10 sq. ft. 56 sq. in. **4.** 5 sq. ft. 96 sq. in.

5. 12 sq. ft. 76 sq. in. **6.** 11s. 11d.

7. 148 c. in. **8.** 130 c. in.

9. (i) 616 c. in., (ii) 2 c. ft. 702 c. in. **10.** 110 c. in.

11. 88 c. in. **12.** 4 c. ft. 1712 c. in.

EXAMPLES. XVIII. B.

1. 5 sq. ft. 96 sq. in. **2.** 222·2 sq. in.

3. 3644·26 sq. in. **4.** 376·99 sq. in.

5. 200·43 sq. in. **6.** 4 in. **7.** 105·65 c. in.

8. 329·09 c. in. **9.** 1575 gallons. **10.** 28·6 c. in.

11. 344·14 c. in. **12.** $667\frac{1}{3}$ c. in.

13. 1 c. ft. 472 c. in. **14.** 66·19 in.

15. 145·49 sq. in. 16. $\frac{n^3-(n-1)^3}{n^3}$. 17. $\frac{h}{\sqrt[3]{2}}$.
18. $2\frac{3}{4}$ gallons. 19. 5·2 pints. 20. 1·6 in.
21. 6 in. 22. 10 in.; 5 in.

EXAMPLES. XIX.

1. 45 c. in. 2. 463 oz. 3. 185 tons.
4. 1284 sq. in. 5. 5773·5 c. in.; 1600 sq. in.
7. $409\frac{1}{3}$ c. in. 8. 3124·8 tons. 9. 609 days.
10. 12 ft., 4 ft.; 498·8 c. ft.; 194 sq. ft.

EXAMPLES. XX. A.

1. (i) 616 sq. in., (ii) $38\frac{1}{2}$ sq. ft., (iii) 273 sq. yds. 7 sq. ft.
2. £23. 2*s*. 3. 264 sq. ft. 4. 55 sq. ft.
5. (i) $3\frac{1}{2}$ in., (ii) 1 ft. 2 in. 6. 4·24 in.
7. (i) $1437\frac{1}{3}$ c. in., (ii) $22\frac{1}{2}\frac{1}{4}$ c. ft., (iii) 38·803 c. in.
8. $735\frac{4}{7}$ oz. 9. 26·[illegible] gallons. 10. 36 gallons nearly.
11. 840. 12. 5. 13. 2592.
14. 368 lbs. 15. $381\frac{1}{3}$ c. in. 16. $268\frac{4}{7}$ c. in.
17. 286 sq. in. 18. 47·74 sq. in. 19. 1945 oz.
20. 549 lbs. 21. 161·7 lbs. 22. 951 oz.
23. (i) 3 in., (ii) 3 ft. 6 in.
24. (i) $1437\frac{1}{3}$ c. in., (ii) 22 c. ft. 792 c. in.
25. (i) 154 sq. in., (ii) $12\frac{4}{7}$ sq. ft. 26. 2 in. 27. $\frac{1}{2}$ in.

EXAMPLES. XX. B.

1. 2299·6 sq. in. 2. 11·31 sq. in
3. (i) 2 : 3, (ii) ·5236 : 1. 4. 104·72 sq. in.
5. 32·48 sq. in. 6. 1 in. 7. 14 oz.
8. 4188·8 c. in. 10. 26·67 cm. 11. $2\frac{1}{2}$ in.
12. 10·27 gallons. 13. 3·3 cm.
14. 2764·6 sq. in.; 25 gallons, nearly. 15. ·4764.
16. $\frac{1}{3}$. 17. 1 ft. 18. 2 ft.
19. 14·88 in. 20. 18·02 cm. 21. 22·6 in.
22. 7·34 in. 23. 1 in. nearly. 24. 1 cm.
25. 3·6 in. 26. 1·1 in. 27. $1\frac{1}{4}$ in. pearl
28. 8 oz. 5 dwts. 29. 2·04 in. 30. 1·37 cm.
31. $1\frac{1}{2}$ in. 32. 1·77 cm.

EXAMPLES. XXI. A.

1. (i) 110 sq. in., (ii) 11 sq. ft. 2. (i) 110 sq. in., (ii) $2\frac{3}{4}$ sq. ft.
3. 44 sq. in. 4. (i) $113\frac{1}{7}$ sq. in., (ii) 7 sq. ft. $98\frac{2}{7}$ sq.
5. 37 sq. ft. 62 sq. in. 6. 45·21 sq. ft.
7. 27 sq. ft. 135 sq. in. 8. (i) $34\frac{5}{6}$ c. in., (ii) 264 c. in.
9. (i) 132 c. in., (ii) $810\frac{2}{7}$ c. in.
10. (i) $469\frac{1}{3}$ c. in., (ii) 2 c. ft. 108 c. in. 11. $154\frac{2}{3}$ c. in.
12. $3358\frac{2}{3}$ c. in. 13. 8125 : 1701.

EXAMPLES. XXI. B.

1. (i) 62·83 sq. in., (ii) 40 sq. ft. 52 sq. in., (iii) 31·416 sq. in.
(iv) 157·28 sq. in. 2. 2011 sq. ft. 3. ·2647
4. Distance of centre, 8 ft. 5. 42 ft. 6. 699 lbs.
7. $\frac{5}{9}$. 8. $\frac{1}{10}$; ·87 in. 9. 13 cm.
11. 904·82 sq. in.; 2412·75 c. in.
12. 1263·1 sq. in. 13. ·1464; ·05806.

EXAMPLES. XXII. A.

1. 16 : 9; 64 : 27. 2. 125 : 8 3. 3·0625.
4. 11 : 9. 5. 15 ft. 4 in. 6. $2\frac{1}{7}$ lbs.
7. 65 in. 8. 75 : 56. 9. $16\frac{1}{3}$ c. in.
10. 2·47 in. 11. 12·23 in. 12. $\sqrt[3]{n-1} : \sqrt[3]{n} : \sqrt[3]{n+1}$.
13. 1 : 7. 14. 1 : 7 : 19.

EXAMPLES. XXII. B.

1. (i) 847 sq. in.; $741\frac{1}{3}$ c. in., (ii) $75\frac{3}{8}$ sq. in.; 33·08 c. in.
(iii) 154·88 sq. in.; 54·21 c. in.
2. 52·8 in.; 15·4 in.; 18·2 in. 3. ·28 in.
4. $4\pi r$; $r(4\pi-1)$. 5. $3\pi : 4$; 9 : 16.

EXAMPLES. XXIII.

1. 43 ac. 1 r. 8 p. 9·7 sq. yds. 2. 21·331 yds.
3. 439·98 ft. 4. 15·925 in. 5. 3 r. 34 p. 22·6 sq. yds.
6. 15958 sq. in. 7. ·60591 ac. = 2 r. 16 p. 28·6 sq. yds.
8. 13·365 in. 9. 147·22 sq. ft. 10. 2626 sq. ft.
11. 30614·7 sq. in. 12. 17625·7 sq. yds. 13. 48° 35′ 25″.

14. 205887 sq. in. 15. 9928·33 miles. 16. 179·882 in.
17. 195099 gallons. 18. 321·253 ft. 19. 100,000 shot.
20. 10250·6 c. in 21. 24914 in. 22. 105·133 in.
23. 42·7569 in 24. 3266120 c. in 25. 33° 0′ 16″.
26. 11198·1 c. in. 27. 54° 41′ 8″. 28. 28° 23′ 18″.
29. 143604 oz. 30. 268·272 gallon 31. 2·17685 in.
32. 54° 16′ 20″. 33. 23·6 in. 34. 3923·08 c. ft.
35. 825·828 c. in 36. 37·5734 in 37. 44° 35′ 47″.
38. 3·79729 in. 39. 70° 31′ 43″ 40. 109° 28′ 17″.
41. 1·77808 in. 42. 10·6212 in.; 1417·61 sq. in.
43. 26° 33′ 54″. 44. 39536·3 c. in.; 4885·38 sq. in.
45. 100 in.; 1° 8′ 44″ 46. 14·84359 in.; 1·94785 in.
47. 70,567 tons 48. 6·48385 in. 49. 304 oz. 11 dwt.
50. 100 times. 51. 415·474 c. in.
52. 801·941 sq. in.; 1795·68 c. in 53. 13059·3 sq. in.; 113097 c. in

EXAMPLES. XXIV.

SECTION I.

A. 1. 1764. 2. 16·06 sq. ft. 3. 20 ft. 4. 40 ft.
5. 68 sq. ft. 6. 282150 gallons. 7. 11319 c. ft.; 3465 sq. ft.
8. 9 ft. 9. 1000. 10. 25 in. 11. 9·9 ft.
12. 6·25, 43·75, 118·75, 231·25 c. ft 13. 10 ft. 14. 12·7 acres.
B. 15. 12 ft. 16. £1. 11s. 17. 250000 gallons.
18. 144 gallons. 19. 19800000 gallons.
20. 10296 sq. ft.; 125 ft.; 82·37 ft. 21. 173·2 c. in.
22. 16·875 in. 23. £5. 11s. 9d. 24. 157·08 sq. ft.
25. 14s. 7d. 26. 150 sq. yds. 3 sq. ft. 108 sq. in.
27. 115·5 sq. in., $205\frac{1}{3}$ sq. in.
28. 8 oz. 3 dwts. 29. 892·92 yds. 30. 614 : 1915.
31. 68·02 sq. in. 32. 8200·84 c. in. 3644·82 sq. in.
33. 7·21721 acres. 34. 32·91 sq. in 35. $\frac{1}{2}$ oz.
36. 31·5 sq. ft. 37. [illegible]8·75 c. ft.
C. 38. 4020. 39. 6 chains; $2\frac{1}{2}$ chains. 40. 3 : 4 : 5.
41. 1254 yds. 42. 24·249. 43. ·[illegible]11 acres.
44. £5. 10s. 45. 468 ft. 46. 2 : 1. 47. 16.
48. 31·05 gallons. 49. 35·39 oz. 50. 125·71 sq. ft.

51. 197·72 c. in. 52. $4\pi d$; $(4\pi - 1)d$.
54. 4200 c. ft. 55. 24·02 c. in. 56. 126 c. ft. 1152 in.
57. 22·48 in. 58. 30·16 c. in.; 52·[illegible] sq. in. 59. 863·89 sq. in.
D. 61. £198. 9*s*. 62. 44359·64 sq. yds. 63. 1869·73.
64. 42 ac. 1 r. 20 p. 65. £10. 10*s*. 11¼*d*. 66. 11581 c. ft. 1332 in.
67. 198956300 sq. mi., nearly; 263883000000 c. mi., roughly.
68. 19 ac. 3 r. 2·8 p. 69. 9 ac. 3 r. 39·58 p. 70. 6 ac. 3 r.
71. 1·375 acres. 72. 6·873 acres. 73. 66350·59 c. in.
74. 4 ft. 75. 48 c. ft. 76. ·52 c. ft.
77. 212 c. ft. 78. 429·63 c. ft. 79. 128·63 c. ft.
80. 2·4[illegible] acres.

Section II.

A. 1. 3·337728, 3·343262; mean = 3·340495. 2. 9·54 tons.
3. 2·534 in. 4. ·52 in. nearly; ·83 in. nearly.
5. 19 : 7 : 1. 6. 7[illegible]·21 yds., 38·105 yds., 30·403 yds.
7. 386425 lbs. = 172 tons 10 cwt. 25 lbs. 8. 6·48 in.
9. 13403 sq. yds. = 2 ac. 3 r. 3 p. 2·25 sq. yds.
10. 784 sq. ft.; 23 ft. nearly. 11. £5. 10*s*.
12. 4·69 ft. 13. 59·55 c. ft. 14. £466. 7*s*. 1*d*.
15. 1·027 sq. in. 16. 643000.
17. 2130·62 sq. ft.; 12016·0 c. ft. 18. 21·7 in.
19. 12 lbs. 6·6 oz. 20. 1612·5 sq. ft.
21. 4·189 c. in.; ·524 c. in. 22. 1·3 ft.; 8·16816 ft. 23. 4 in.
24. 933·59 c. in.; 1581·70 c. in. 25. £[illegible]33. 17*s*. 3*d*.
26. 573[illegible] in. 27. 37·7 sq. in., nearly. 28. 56·57 sq. in.
29. 143 yds. 30. 10·5 in. 31. 950 sq. yds.
32. 9·42 c. ft. 33. 93·98 sq. in. 34. 1889·67 sq. ft.
35. 92·4 sq. in.; 14·941 in. 36. 20 ft. 37. 1·24 in.
38. 1 in. 39. 183·26 sq. in. 40. 13 in. nearly.
42. 39·37 ft. per minute. 43. 3·36 ft. 44. 4·06293 in.
45. 1·819 ft.; 9 in. 46. 14084 grammes. 47. 8·29 in.; 237·[illegible]7 sq. in.
48. 225 nearly; 113·1 sq. in. 49. 1856 oz.
50. 119·7299 c. in.; 838·1091 c. in.; 402·6256 sq. in.; 307·8768 sq. in.
B. 51. 233·509 ft.; 239 tons. 52. 165·748 c. yds. 53. 960·08 sq. ft.
54. 19721·4 oz. 55. 101543·26 sq. ft.; 1129·61 ft.
56. ⅔ of height of vessel. 57. 22·6544 sq. yds.
58. 2¾ in. nearly; diameters 3·4648 in.; 4·4097 in.; height 5·6096 in.

59. 133·785 ft. 60. ·292187 tons. 61. £1996*l.* 18*s.* 5*d.*

62. 1221·88 c. in. 63. 96 c. in.; 138·528 sq. in.

64. 2·33631 tons. 65. 209·58 sq. yds. 66. 13·2698 sq. in.

67. 190·765 sq. ft. 68. 7·432 ft. 69. 1 ft.

70. 27·713 sq. in.; 7·542 c. in. 71. 151·74 c. ft.

72. 382674 sq. miles nearly. 73. 114582 sq. ft.

74. 4·0825 in. 75. π sq. ft.

76. 12·1382 sq. in. 77. 15 ft.

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